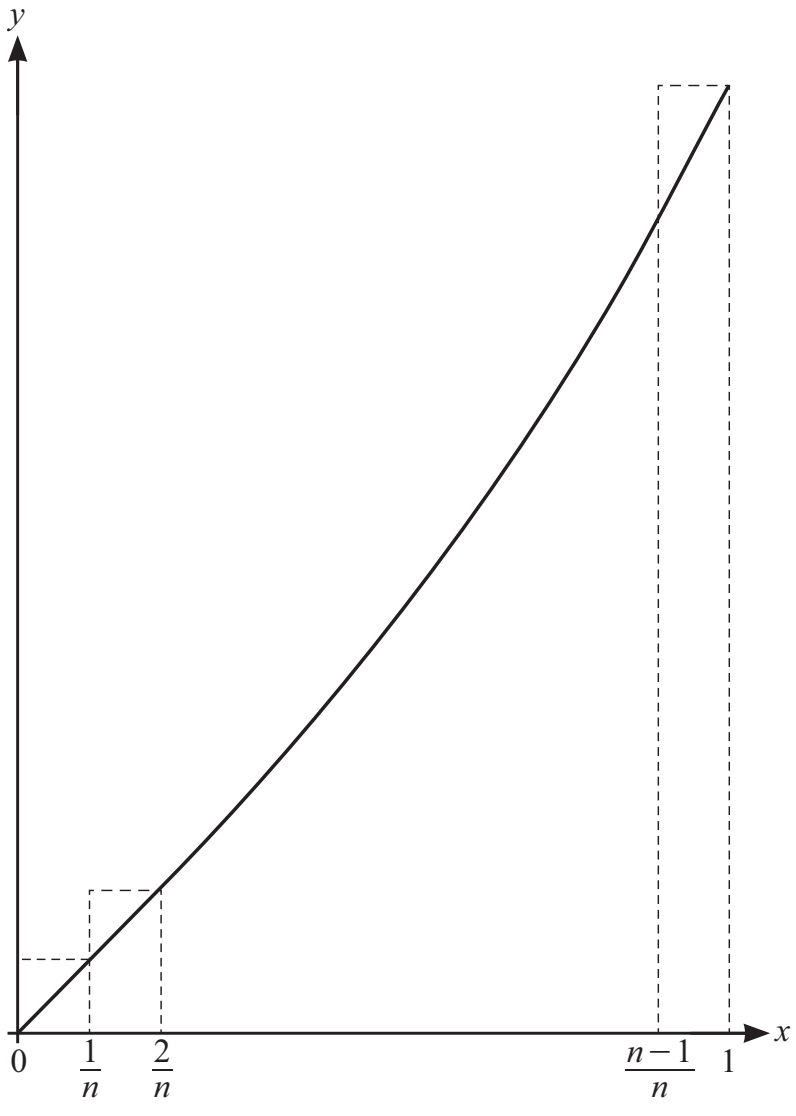
5



The diagram shows the curve with equation $y = \frac{1}{3}x^3 + x$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \left(\frac{1}{3}x^3 + x\right)\mathrm{d}x < U_n$, where

$$U_n = \frac{1}{12}\left(1 + \frac{1}{n}\right)\left(7 + \frac{1}{n}\right). \qquad [5]$$

(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 \left(\frac{1}{3}x^3 + x\right)\mathrm{d}x$. [4]

(c) Show that $\lim_{n \to \infty}(U_n - L_n) = 0$. [2]