

Algebra and graphs

IGCSE Mathematics

This handout covers Topic 2, Algebra and graphs. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

Working with algebra

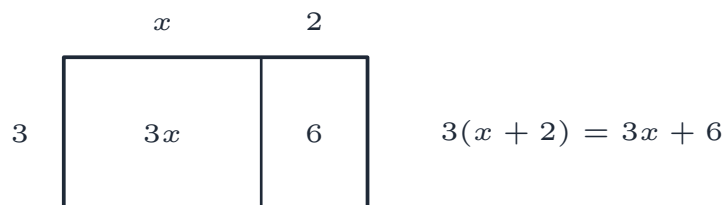
In **algebra** 代数 we use letters to stand for numbers. A letter whose value can change is a **variable** 变量. To **substitute** 代入 means to put a number in place of a letter.

Worked example. Find the value of $3x^2 - 2y$ when $x = 4$ and $y = 5$.

$$3 \times 4^2 - 2 \times 5 = 3 \times 16 - 10 = 48 - 10 = 38.$$

Simplifying and expanding

A **term** 项 is a single part of an **expression** 表达式, such as $5a$ or $-9b$. **Like terms** 同类项 have exactly the same letters; you may add or subtract them. The number in front of the letter is the **coefficient** 系数.



Expanding a bracket with an area model

$$\begin{array}{c} 3a + 2a + 4b - b \\ \downarrow \text{collect like terms} \\ 5a + 3b \end{array}$$

Collecting like terms: add terms with the same letter

Worked example. Simplify $2a^2 + 3ab - 1 + 5a^2 - 9ab + 4$.

Collect like terms: $2a^2 + 5a^2 = 7a^2$, $3ab - 9ab = -6ab$, $-1 + 4 = 3$. So the answer is

$$7a^2 - 6ab + 3.$$

To **expand** 展开 means to multiply out **brackets** 括号. Multiply every term inside by the term outside; for two brackets, multiply every term in the first by every term in the second.

Worked examples.

$$3x(2x - 4y) = 6x^2 - 12xy.$$

$$(2x + 1)(x - 4) = 2x^2 - 8x + x - 4 = 2x^2 - 7x - 4.$$

For three brackets (**Extended**), expand two first, then multiply by the third:

$$(x - 2)(x + 3)(2x + 1) = (x^2 + x - 6)(2x + 1) = 2x^3 + 3x^2 - 11x - 6.$$

Factorising

To **factorise** 因式分解 is the opposite of expanding: write the expression as a product of brackets. Always take out the **common factor** 公因式 first.

Worked example. $9x^2 + 15xy = 3x(3x + 5y)$, because $3x$ divides both terms.

The following patterns are **Extended**.

Grouping (four terms): take a common factor from each pair.

$$xy + 2x + 3y + 6 = x(y + 2) + 3(y + 2) = (x + 3)(y + 2).$$

Difference of two squares 平方差: $a^2 - b^2 = (a + b)(a - b)$.

$$9x^2 - 16 = (3x + 4)(3x - 4).$$

Perfect square 完全平方: $a^2 + 2ab + b^2 = (a + b)^2$.

$$x^2 + 6x + 9 = (x + 3)^2.$$

Quadratic 二次 expressions $ax^2 + bx + c$: find two numbers that multiply to $a \times c$ and add to b , then split the middle term.

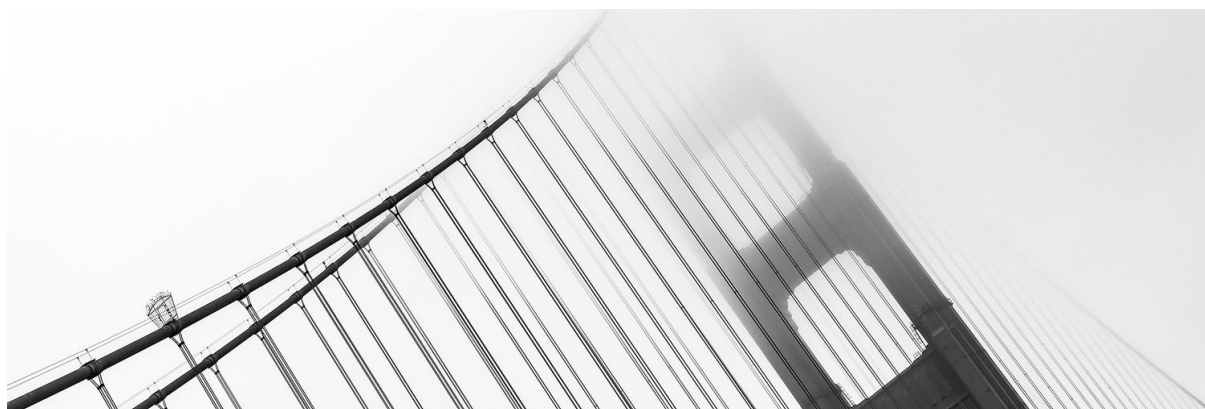
Worked example. Factorise $2x^2 + 7x + 3$.

Here $a \times c = 6$ and $b = 7$. The numbers 1 and 6 work. Split and group:

$$2x^2 + x + 6x + 3 = x(2x + 1) + 3(2x + 1) = (x + 3)(2x + 1).$$

For $ax^3 + bx^2 + cx$, take out the common x first: $2x^3 + 7x^2 + 3x = x(2x^2 + 7x + 3) = x(x + 3)(2x + 1)$.

Completing the square (Extended)



A suspension bridge cable hangs in a parabola—the graph of a quadratic.

Image: Dietmar Rabich, CC BY-SA 4.0 (commons.wikimedia.org)

Completing the square 配方法 rewrites $x^2 + bx + c$ as $(x + p)^2 + q$. Take half of the x -coefficient, square it, then balance.

Worked example. Write $x^2 + 6x + 1$ in completed-square form.

Half of 6 is 3, and $3^2 = 9$:

$$x^2 + 6x + 1 = (x + 3)^2 - 9 + 1 = (x + 3)^2 - 8.$$

When the x^2 has a coefficient, take it out of the first two terms first:

$$2x^2 + 8x + 3 = 2(x^2 + 4x) + 3 = 2((x + 2)^2 - 4) + 3 = 2(x + 2)^2 - 5.$$

Algebraic fractions (Extended)

An **algebraic fraction** 分式 has algebra on the top or bottom. Add and subtract using a common denominator; multiply and divide as with ordinary fractions.

Worked examples.

$$\frac{x}{3} + \frac{x-4}{2} = \frac{2x}{6} + \frac{3(x-4)}{6} = \frac{2x+3x-12}{6} = \frac{5x-12}{6}.$$

$$\frac{3a}{4} \div \frac{9a}{10} = \frac{3a}{4} \times \frac{10}{9a} = \frac{30a}{36a} = \frac{5}{6}.$$

To simplify a **rational expression** 有理式, factorise the top and bottom, then cancel common brackets.

$$\frac{x^2 - 2x}{x^2 - 5x + 6} = \frac{x(x-2)}{(x-2)(x-3)} = \frac{x}{x-3}.$$

Indices in algebra

The laws of **indices** 指数 work with letters too: $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, and $(a^m)^n = a^{mn}$.

Worked examples.

$$(5x^3)^2 = 25x^6, \quad 12a^5 \div 3a^{-2} = 4a^7, \quad 6x^7y^4 \times 5x^{-5}y = 30x^2y^5.$$

You can also solve simple index equations by writing both sides with the same **base** 底数.

Worked example. Solve $2^x = 32$. Since $32 = 2^5$, you get $x = 5$.

Equations

An **equation** 方程 says two expressions are equal. To solve a **linear** 一次 equation, do the same operation to both sides until the **unknown** 未知数 is alone.

$$2x + 3 = 11 \quad (\text{start})$$

$$2x = 8 \quad \text{subtract 3 from both sides}$$

$$x = 4 \quad \text{divide both sides by 2}$$

Solve by doing the same to both sides

Worked example. Solve $5 - 2x = 3(x + 7)$.

$$5 - 2x = 3x + 21 \Rightarrow 5 - 21 = 3x + 2x \Rightarrow -16 = 5x \Rightarrow x = -\frac{16}{5}.$$

Fractional equations (Extended)

A **fractional equation** 分式方程 has the unknown in a denominator. Multiply both sides by the denominator to clear it.

Worked example. Solve $\frac{x}{2x+1} = 4$.

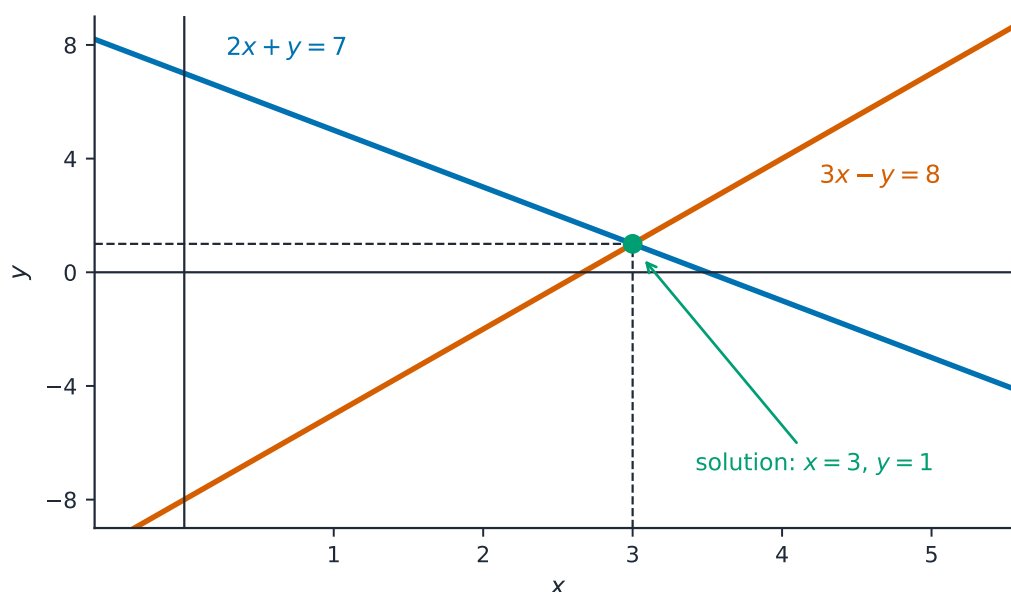
$$x = 4(2x+1) = 8x+4 \Rightarrow -7x = 4 \Rightarrow x = -\frac{4}{7}.$$

Simultaneous equations

Simultaneous equations 联立方程 are two equations solved together. For two linear equations, add or subtract to remove one letter.

Worked example. Solve $2x + y = 7$ and $3x - y = 8$.

Adding removes y : $5x = 15$, so $x = 3$. Then $y = 7 - 2(3) = 1$.



The solution of simultaneous equations is where their graphs cross

For one linear and one **quadratic** equation (**Extended**), substitute the linear into the curve.

Worked example. Solve $y = x + 2$ and $y = x^2$.

$$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0,$$

so $x = 2$ (giving $y = 4$) or $x = -1$ (giving $y = 1$).

Solving quadratic equations (Extended)

There are three methods.

- **By factorising:** $x^2 + 5x + 6 = 0 \Rightarrow (x + 2)(x + 3) = 0 \Rightarrow x = -2$ or $x = -3$.
- **By completing the square:** $x^2 + 6x + 1 = 0 \Rightarrow (x + 3)^2 = 8 \Rightarrow x + 3 = \pm 2\sqrt{2} \Rightarrow x = -3 \pm 2\sqrt{2}$.
- **By the quadratic formula** 求根公式, which is given in the exam:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Worked example (formula). Solve $2x^2 + 3x - 1 = 0$. Here $a = 2$, $b = 3$, $c = -1$:

$$x = \frac{-3 \pm \sqrt{9 + 8}}{4} = \frac{-3 \pm \sqrt{17}}{4}.$$

Changing the subject

To **change the subject** 公式变形 of a formula means to rearrange it so a chosen letter is alone on one side.

Worked example. Make r the subject of $A = \pi r^2$ (**Extended**, because of the power).

$$r^2 = \frac{A}{\pi} \Rightarrow r = \sqrt{\frac{A}{\pi}}.$$

When the letter appears twice (**Extended**), collect those terms and factorise. To make x the subject of $y = \frac{x+1}{x-1}$:

$$y(x-1) = x+1 \Rightarrow yx - x = 1+y \Rightarrow x(y-1) = 1+y \Rightarrow x = \frac{1+y}{y-1}.$$

Inequalities

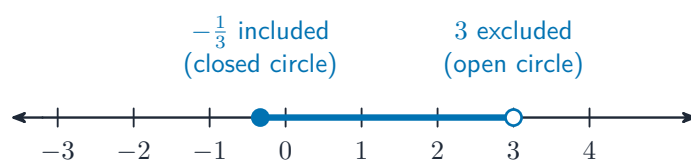
An **inequality** 不等式 uses $<$, $>$, $\leq$ or $\geq$. Solve it like an equation, but **reverse the sign** if you multiply or divide by a negative number.

Worked example. Solve $-3 \leq 3x - 2 < 7$.

Add 2 to all parts, then divide by 3:

$$-1 \leq 3x < 9 \Rightarrow -\frac{1}{3} \leq x < 3.$$

On a **number line** 数轴, use an open circle for $<$ or $>$ (value not included) and a closed circle for $\leq$ or $\geq$ (value included).



$-\frac{1}{3} \leq x < 3$: a closed circle includes the end value, an open circle excludes it.

Regions (Extended)

An inequality in two letters describes a **region** 区域 of the graph. Draw the boundary line (broken for $<$ or $>$, solid for $\leq$ or $\geq$) and shade the unwanted side. You may also be asked to list the inequalities that define a given region.

Sequences



Romanesco broccoli: self-similar spirals form a natural number pattern.

Image: Jon Sullivan, Public domain (commons.wikimedia.org)

A **sequence** 数列 is a list of numbers that follow a rule. The **term-to-term rule** 递推规则 tells you how to get the next term from the one before.

To find the rule for the term in position n (the n th **term**), look at how the terms change.

Linear sequence (the terms go up by the same amount each time). The difference is the multiple of n .

Worked example. Find the n th term of 2, 5, 8, 11, ...

The terms go up by 3, so start with $3n$. Since $3 \times 1 = 3$ but the first term is 2, subtract 1: the n th term is $3n - 1$.

Quadratic sequence (the differences themselves change by the same amount). The second difference equals $2 \times$ the coefficient of n^2 .

Worked example. Find the n th term of 2, 5, 10, 17, ...

First differences are 3, 5, 7; the second difference is 2, so the n^2 part is $1n^2$. Subtracting n^2 (1, 4, 9, 16) from the sequence leaves 1, 1, 1, 1. So the n th term is $n^2 + 1$.

A **cubic** 三次 sequence such as 1, 8, 27, 64, ... has n th term n^3 . An **exponential sequence** 指数数列 such as 2, 6, 18, 54, ... multiplies by a fixed number each time; here the n th term is $2 \times 3^{n-1}$.

Direct and inverse proportion (Extended)

Two quantities are in **direct proportion** 正比例 if one is always a fixed multiple of the other: $y \propto x$ means $y = kx$, where k is a **constant** 常数. They are in **inverse**

proportion 反比例 if one rises as the other falls: $y \propto \frac{1}{x}$ means $y = \frac{k}{x}$. The symbol $\propto$ is read "is proportional to". You can also have proportion to a square, square root, cube or cube root.

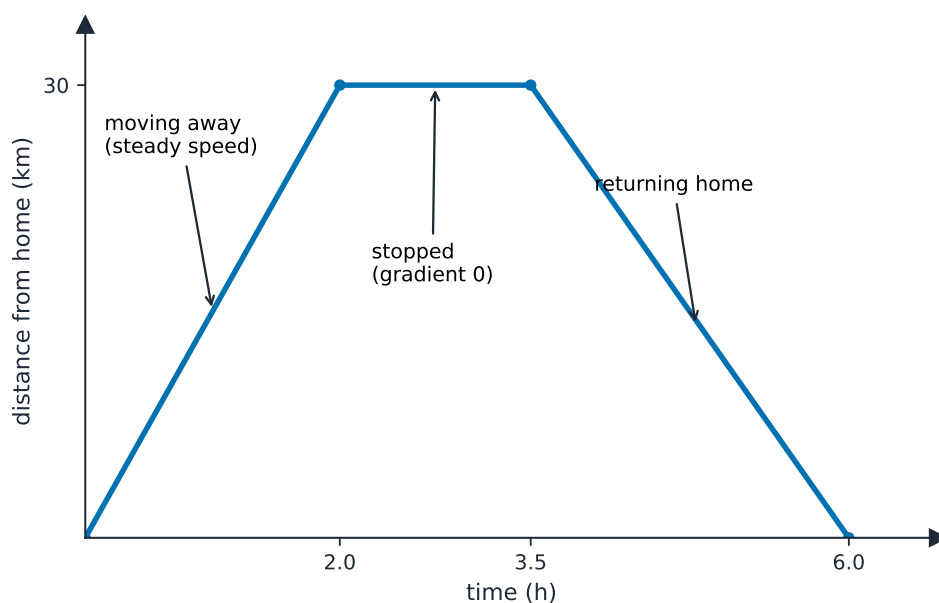
Worked example. y is in direct proportion to x , and $y = 12$ when $x = 3$. Find y when $x = 7$.

First find k : $12 = k \times 3$, so $k = 4$ and $y = 4x$. Then $y = 4 \times 7 = 28$.

Graphs in practical situations

The **gradient** 斜率 (steepness) of a graph shows a **rate of change** 变化率.

- A **travel graph** 行程图 (distance–time graph) has gradient equal to speed; a flat part means the object is not moving.
- A **conversion graph** 换算图 is a straight line used to change between two units (for example, miles and kilometres).



On a distance–time graph the gradient is the speed; a flat section means the object has stopped.

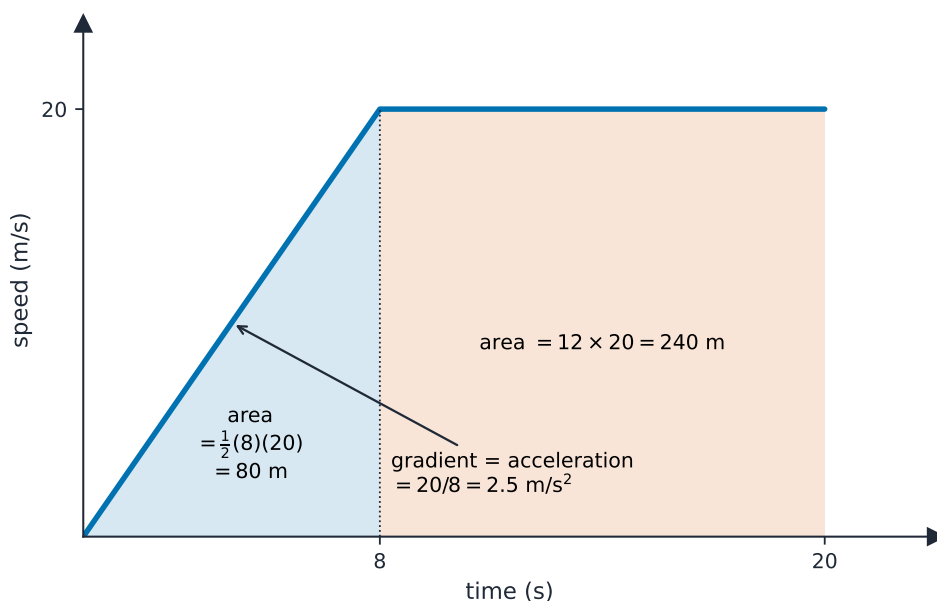
Speed–time graphs (Extended)

On a speed–time graph the gradient is the **acceleration** 加速度 (or **deceleration** 减速 if the speed falls), and the **area** 面积 under the graph is the **distance** 距离 travelled.

Worked example. A car speeds up from rest to 20 m/s in 8 s, then stays at 20 m/s for 12 s. Find the acceleration and the total distance.

Acceleration = $\frac{20}{8} = 2.5 \text{ m/s}^2$. The distance is the area: a triangle plus a rectangle,

$$\frac{1}{2} \times 8 \times 20 + 12 \times 20 = 80 + 240 = 320 \text{ m.}$$



On a speed–time graph the gradient is the acceleration and the area underneath is the distance travelled.

Graphs of functions

To draw a graph, make a **table of values** 数值表: choose values of x , work out y , then plot the **coordinates** 坐标 and join them with a smooth curve.

The points where a graph crosses the x -axis (the horizontal **axis** 坐标轴) are the **roots** 根—the solutions of $y = 0$.

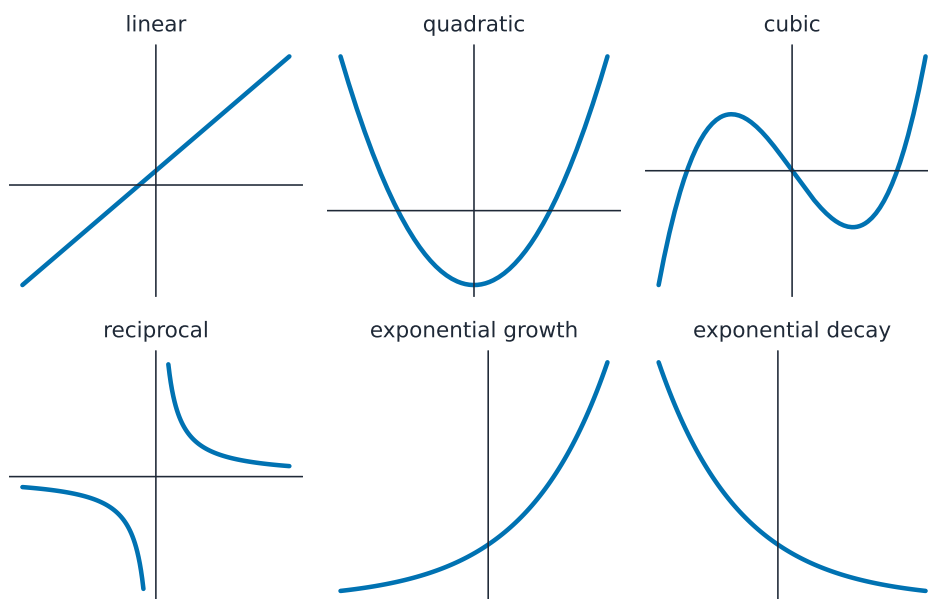
You can solve an equation by reading a graph. The **intersection point** 交点 of a line and a curve gives the solution of the two equations together.

For **exponential growth** 指数增长 and **exponential decay** 指数衰减, the graph of $y = a b^x + c$ rises (or falls) faster and faster and flattens towards a horizontal line.

Sketching curves

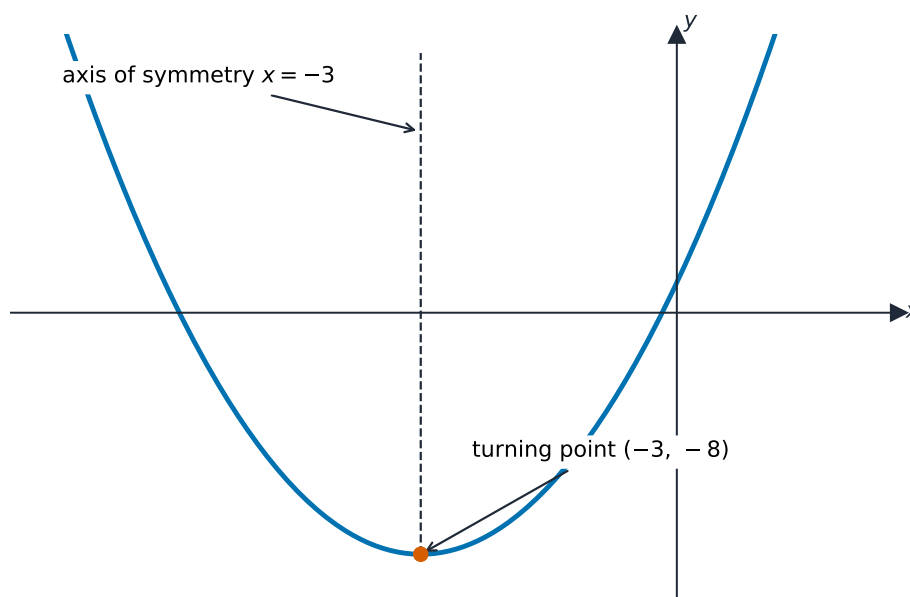
A quick sketch should show the right shape and the key features: where it crosses the axes, any **symmetry** 对称, and any line the curve gets close to.

Function	Shape
linear, $y = mx + c$	straight line; gradient m , y-intercept 截距 c
quadratic, $y = ax^2 + bx + c$	a parabola 抛物线 (U-shape if $a > 0$, $\cap$ -shape if $a < 0$)
cubic, $y = ax^3 + bx + c$	an S-shaped curve
reciprocal, $y = \frac{a}{x} + b$	two separate curves
exponential, $y = ar^x + b$	fast growth or decay



The basic graph shapes; knowing each shape lets you sketch quickly from the equation.

For a parabola, completing the square gives the **turning point** 转折点 (the lowest or highest point). For example $y = (x + 3)^2 - 8$ has its turning point at $(-3, -8)$.

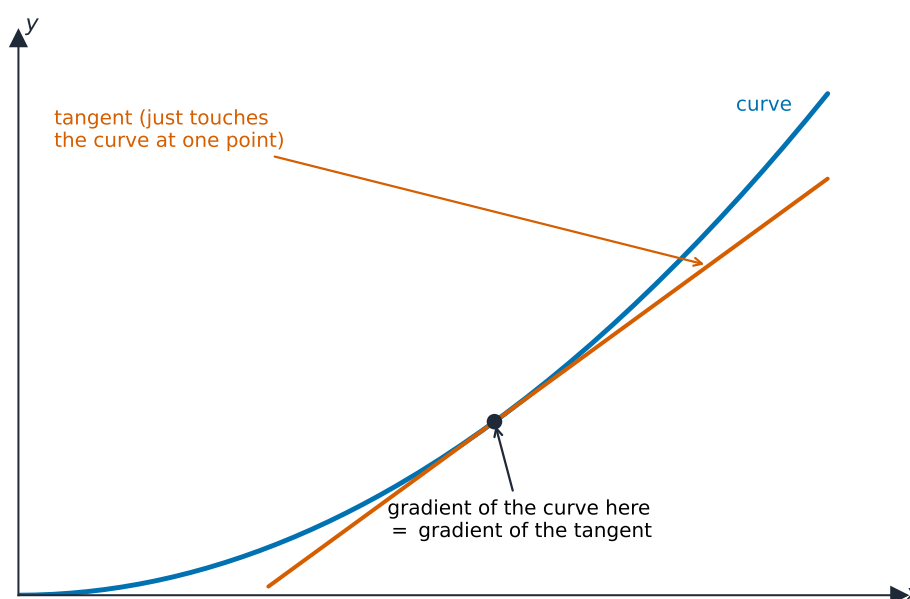


Completing the square, $y = (x + 3)^2 - 8$, shows the turning point $(-3, -8)$ and the line of symmetry $x = -3$.

An **asymptote** 渐近线 is a line that the curve gets closer and closer to but never touches—for example, the x -axis for $y = \frac{a}{x}$, or the line $y = b$ for $y = ar^x + b$.

Differentiation (Extended)

Differentiation 微分 finds the gradient of a curve at any point. You can estimate it by drawing a **tangent** 切线 (a line that just touches the curve) and measuring its gradient.



The gradient of a curve at a point equals the gradient of the tangent there—what differentiation finds.

The exact rule: if $y = ax^n$, then the **derivative** 导数 is

$$\frac{dy}{dx} = a n x^{n-1}.$$

Differentiate a sum term by term.

Worked example. If $y = x^3 + 2x^2 - 5x$, then $\frac{dy}{dx} = 3x^2 + 4x - 5$.

A **stationary point** 驻点 (turning point) is where the gradient is zero, so set $\frac{dy}{dx} = 0$.

Worked example. Find the turning point of $y = x^2 - 6x + 5$.

$$\frac{dy}{dx} = 2x - 6 = 0 \Rightarrow x = 3, \quad y = 3^2 - 6(3) + 5 = -4.$$

The turning point is $(3, -4)$. To decide whether a turning point is a **maximum** 最大值 or a **minimum** 最小值, check the sign of the gradient on each side, or use the second derivative (positive means a minimum).

Functions (Extended)

A **function** 函数 turns each input into one output. We write $f(x)$, for example $f(x) = 3x - 5$, so $f(2) = 1$. The set of allowed inputs is the **domain** 定义域; the set of possible outputs is the **range** 值域.

The **inverse function** 反函数 $f^{-1}(x)$ undoes the function. To find it, write $y = f(x)$, swap the roles, and make x the subject.

Worked example. Find the inverse of $f(x) = 3x - 5$.

$$y = 3x - 5 \Rightarrow x = \frac{y + 5}{3}, \quad \text{so} \quad f^{-1}(x) = \frac{x + 5}{3}.$$

$f(x) = 3x - 5$: do $\times 3$, then -5



$f^{-1}(x) = \frac{x + 5}{3}$: **reverse the order, undo each step**



the inverse takes the output back to the input: $f(2) = 1$, so $f^{-1}(1) = 2$

A function as a machine; the inverse runs it backwards - reverse the order, undo each step

A **composite function** 复合函数 applies one function after another: $gf(x)$ means "do f first, then g ".

Worked example. If $f(x) = 2x$ and $g(x) = x + 3$, then

$$gf(x) = g(2x) = 2x + 3, \quad fg(x) = f(x + 3) = 2(x + 3) = 2x + 6.$$

Exam tips

- When you **expand brackets**, multiply every term and watch the signs, especially with a minus in front: $-(x - 3) = -x + 3$.
- To solve an equation, do the same thing to both sides. When you multiply or divide an **inequality** by a negative number, *flip* the sign.
- **Factorise fully**: take out the highest common factor first, then look for a difference of two squares or a quadratic pattern.
- A **quadratic** usually has two solutions —give both. Check whether the question wants factorising, the formula, or completing the square.
- When substituting into a formula, put each value in brackets first, so signs and powers come out right.