

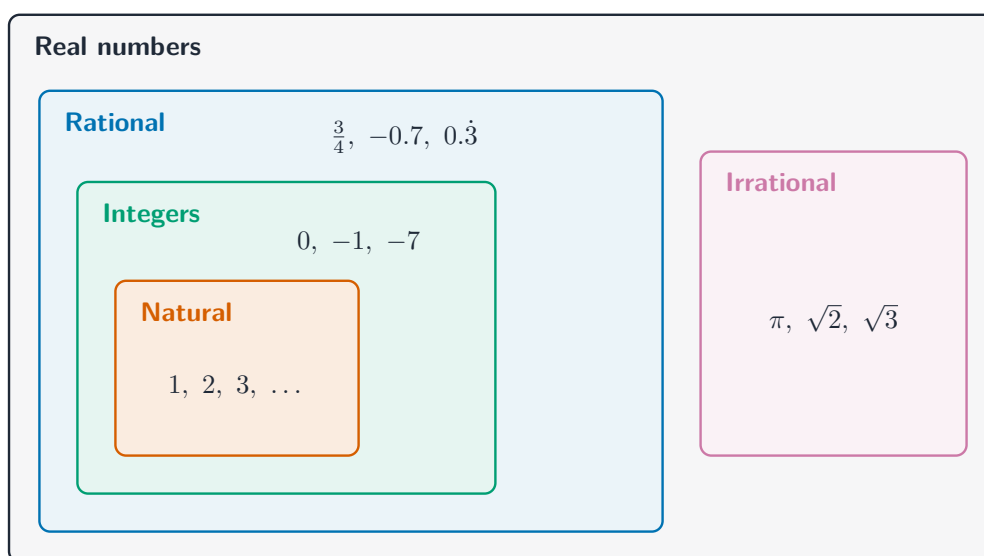
Number

IGCSE Mathematics

This handout covers Topic 1, Number. Cambridge Maths has two levels: Core and Extended. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

Types of number

You must know the words for the different kinds of number. Examiners give marks for using them correctly.



Each set sits inside the next: every natural number is an integer, every integer is rational, every rational is real. Irrational numbers are real but not rational.

Counting numbers and integers

- **natural numbers** 自然数—the counting numbers 1, 2, 3, 4, ...
- **integers** 整数—whole numbers, positive, negative or zero: ..., -2, -1, 0, 1, 2, ...

Factors and multiples

- A **factor** 因数 of a number divides into it exactly, leaving no **remainder** 余数. The factors of 18 are 1, 2, 3, 6, 9, 18.
- A **multiple** 倍数 of a number is that number times an integer. Multiples of 6 are 6, 12, 18, 24, ...
- A **common factor** 公因数 of two numbers is a factor of both.
- A **common multiple** 公倍数 of two numbers is a multiple of both.

Prime, square and cube numbers

- A **prime number** 质数 has exactly two factors: 1 and itself. The first primes are 2, 3, 5, 7, 11, 13, ... Note that 1 is **not** prime.
- A **square number** 平方数 is a whole number times itself: 1, 4, 9, 16, 25, ...
- A **cube number** 立方数 uses a whole number three times: 1, 8, 27, 64, ...

Rational, irrational and reciprocal

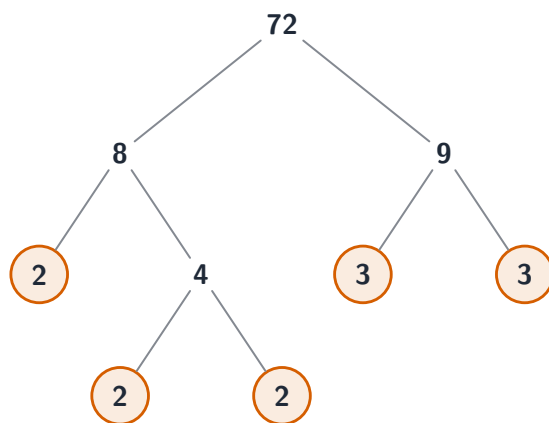
- A **rational number** 有理数 can be written as a **fraction** 分数 $\frac{a}{b}$ of two integers. Examples: $\frac{3}{4}$, 5, 0.7.
- An **irrational number** 无理数 cannot be written this way. Examples: π and $\sqrt{2}$.
- The **reciprocal** 倒数 of a number is 1 divided by that number. The reciprocal of 4 is $\frac{1}{4}$; the reciprocal of 0.25 is 4; the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

Prime factors, HCF and LCM

Every integer above 1 is prime, or can be written as a **product** 乘积 of prime numbers. To write a number as a product of its **prime factors** 质因数, keep dividing by the smallest prime that fits.

Worked example. Write 72 as a product of its prime factors.

$$72 = 2 \times 36 = 2 \times 2 \times 18 = 2 \times 2 \times 2 \times 9 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2.$$



$$72 = 2^3 \times 3^2$$

Keep splitting until every branch ends on a prime (circled); collecting them gives

$$72 = 2^3 \times 3^2.$$

The **highest common factor (HCF)** 最大公因数 of two numbers is the largest factor they share. The **lowest common multiple (LCM)** 最小公倍数 is the smallest multiple they share. Prime factors give a quick method.

Worked example. Find the HCF and LCM of 72 and 120.

First write each as a product of primes:

$$72 = 2^3 \times 3^2, \quad 120 = 2^3 \times 3 \times 5.$$

- HCF: take the **lowest** power of each prime that appears in *both*: $2^3 \times 3 = 24$.
- LCM: take the **highest** power of *every* prime that appears: $2^3 \times 3^2 \times 5 = 360$.

Sets and Venn diagrams

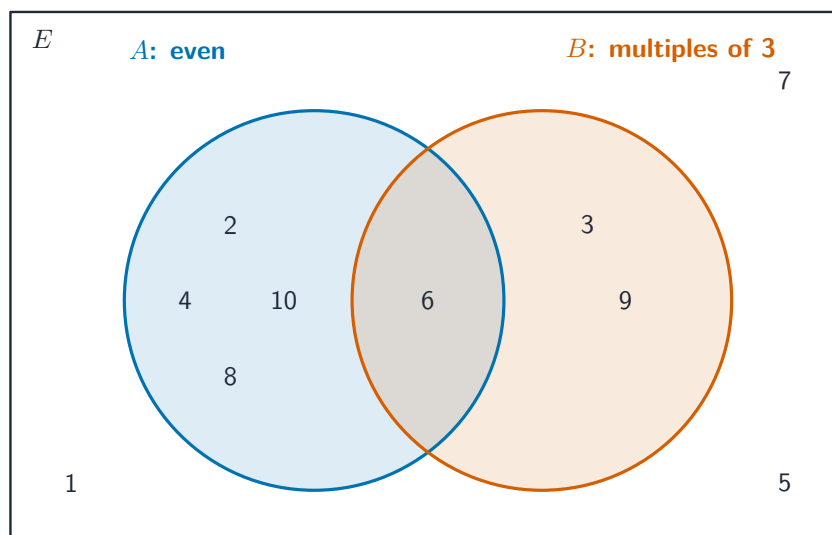
A **set** 集合 is a collection of objects. Each object in the set is an **element** 元素 of the set. You should know this notation:

Symbol	Meaning
$n(A)$	the number of elements in set A
$x \in A$	x is an element of A
$x \notin A$	x is not an element of A
E	the universal set 全集—everything being talked about
A'	the complement 补集 of A —everything not in A
$\emptyset$	the empty set 空集—a set with no elements
$A \subseteq B$	A is a subset 子集 of B —every element of A is also in B
$A \cup B$	the union 并集—elements in A or B or both
$A \cap B$	the intersection 交集—elements in both A and B

A **Venn diagram** 维恩图 draws each set as a circle inside a rectangle (the universal set). Core uses two sets; Extended may use three.

Worked example. $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{\text{even numbers}\}$, $B = \{\text{multiples of 3}\}$.

- $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 6, 9\}$.
- $A \cap B = \{6\}$ —the only number in both.
- $A \cup B = \{2, 3, 4, 6, 8, 9, 10\}$ —the numbers in either set.
- $n(A \cup B) = 7$.



$A \cap B = \{6\}$ is the only number in both circles; $A \cup B$ is everything inside either circle.

You may also see a set written as a rule, e.g. $C = \{x : 1 \leq x \leq 5\}$ means "all values x such that $1 \leq x \leq 5$ ".

Powers and roots

- A **power** 幂 (also called an **index** 指数, plural *indices*) shows how many times to multiply a number by itself: $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$.
- A **square root** 平方根 of a number gives that number when squared: $\sqrt{169} = 13$ because $13^2 = 169$.
- A **cube root** 立方根 works the same way for cubes: $\sqrt[3]{8} = 2$ because $2^3 = 8$.

You should be able to recall the squares from 1^2 to 15^2 (and their roots), and the cubes of 1, 2, 3, 4, 5 and 10.

Worked example. Work out $5^2 \times \sqrt[3]{8}$.

$$5^2 \times \sqrt[3]{8} = 25 \times 2 = 50.$$

The laws of indices

When you multiply or divide powers of the **same base** 底数, use these rules:

$$a^m \times a^n = a^{m+n}, \quad a^m \div a^n = a^{m-n}, \quad (a^m)^n = a^{mn}.$$

Some special powers:

$$a^0 = 1, \quad a^{-n} = \frac{1}{a^n}, \quad a^{\frac{1}{n}} = \sqrt[n]{a}, \quad a^{\frac{m}{n}} = (\sqrt[n]{a})^m.$$

(Fractional powers like $a^{\frac{m}{n}}$ are **Extended**.)

Worked examples.

- $2^{-3} \times 2^4 = 2^{-3+4} = 2^1 = 2.$
- $(2^3)^2 = 2^6 = 64.$
- $2^3 \div 2^4 = 2^{3-4} = 2^{-1} = \frac{1}{2}.$
- $7^{-2} = \frac{1}{7^2} = \frac{1}{49}.$
- $81^{\frac{1}{2}} = \sqrt{81} = 9.$
- $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}.$

multiply the powers

$$a^m \times a^n = a^{m+n}$$

add:

$$2^{-3} \times 2^4 = 2^1 = 2$$

divide the powers

$$a^m \div a^n = a^{m-n}$$

subtract:

$$2^3 \div 2^4 = 2^{-1} = \frac{1}{2}$$

power of a power

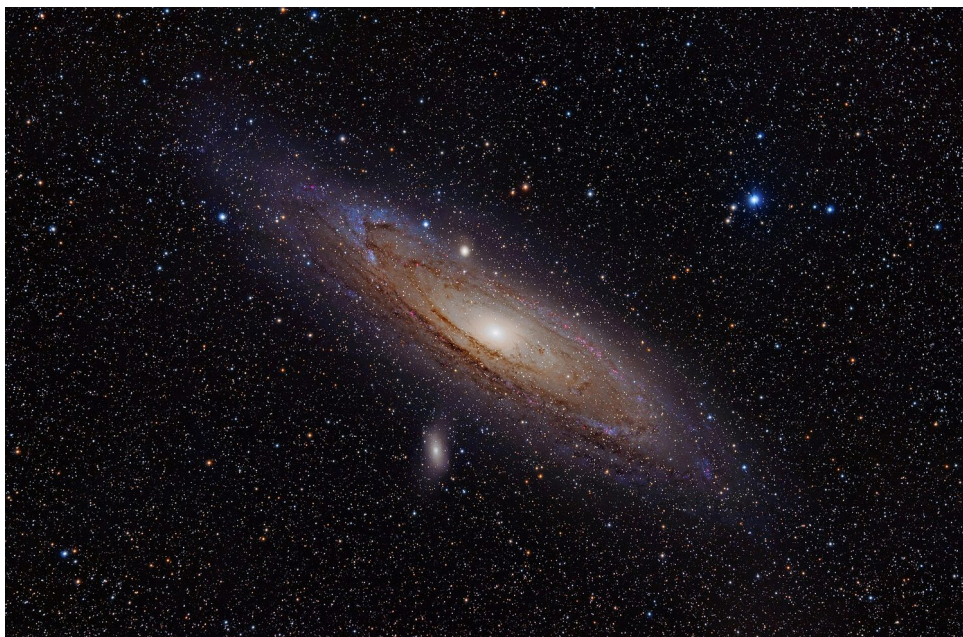
$$(a^m)^n = a^{mn}$$

multiply:

$$(2^3)^2 = 2^6 = 64$$

The three index laws: multiply adds the powers, divide subtracts them, a power of a power multiplies them

Standard form



A galaxy: huge distances are written compactly in standard form.

Image: Adam Evans, CC BY 2.0 (commons.wikimedia.org)

Standard form 科学记数法 writes a number as $A \times 10^n$, where $1 \leq A < 10$ and n is an integer. It is used for very large or very small numbers.

To convert, count how many places the decimal point moves:

- $4\,500\,000 = 4.5 \times 10^6$ —the point moves 6 places left, so the power is positive.
- $0.00072 = 7.2 \times 10^{-4}$ —the point moves 4 places right, so the power is negative.

Worked example. Work out $(3 \times 10^5) \times (2 \times 10^{-2})$.

Multiply the front numbers and add the powers:

$$3 \times 2 = 6, \quad 10^5 \times 10^{-2} = 10^3, \quad \text{so the answer is } 6 \times 10^3.$$

Surds (Extended)

A **surd** 根式 is a root that is irrational, such as $\sqrt{5}$. Leave it in exact form instead of rounding. Two useful rules:

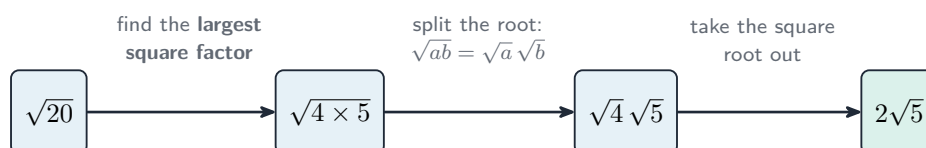
$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}, \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Simplify a surd by taking out the largest square factor.

Worked example. Simplify $\sqrt{20}$ and $\sqrt{200} - \sqrt{32}$.

$$\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \sqrt{5} = 2\sqrt{5}.$$

$$\sqrt{200} - \sqrt{32} = \sqrt{100 \times 2} - \sqrt{16 \times 2} = 10\sqrt{2} - 4\sqrt{2} = 6\sqrt{2}.$$



picking a smaller square factor ($\sqrt{20} = \sqrt{2 \times 10}$) gets stuck -
always take out the **largest** square: 4, 9, 16, 25, ...

Simplifying a surd: take out the largest square factor

To **rationalise the denominator** 分母有理化 means to remove a surd from the bottom of a fraction (the **denominator** 分母). Multiply the top and bottom by a value that clears the surd.

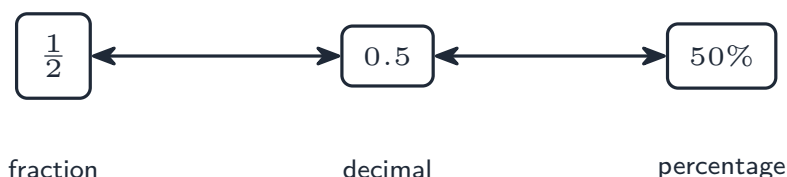
Worked example. Rationalise $\frac{10}{\sqrt{5}}$ and $\frac{1}{-1 + \sqrt{3}}$.

$$\frac{10}{\sqrt{5}} = \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}.$$

$$\frac{1}{-1 + \sqrt{3}} = \frac{1}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = \frac{\sqrt{3} + 1}{3 - 1} = \frac{1 + \sqrt{3}}{2}.$$

Fractions, decimals and percentages

A fraction has a **numerator** 分子 (the top) and a **denominator** (the bottom).



One value three ways: a fraction, a decimal and a percentage

- **proper fraction** 真分数: numerator smaller than denominator, e.g. $\frac{3}{4}$.
- **improper fraction** 假分数: numerator the same or larger, e.g. $\frac{7}{4}$.
- **mixed number** 带分数: a whole number plus a fraction, e.g. $1\frac{3}{4}$.

Change between improper and mixed: $\frac{7}{4} = 1\frac{3}{4}$ because $7 \div 4 = 1$ remainder 3.

A **decimal** 小数 uses place value after a point. A **percentage** 百分比 means "out of 100", so $37\% = \frac{37}{100} = 0.37$.

Converting between forms

To change	Method	Example
fraction $\rightarrow$ decimal	divide top by bottom	$\frac{3}{8} = 3 \div 8 = 0.375$
decimal $\rightarrow$ percentage	multiply by 100	$0.07 = 7\%$
percentage $\rightarrow$ fraction	put over 100, then simplify	$7\% = \frac{7}{100}$
percentage $\rightarrow$ decimal	divide by 100	$34\% = 0.34$

Write a fraction in its **simplest form** 最简形式 by dividing the top and bottom by their HCF: $\frac{18}{24} = \frac{3}{4}$ (both divided by 6).

Recurring decimals (Extended)

A **recurring decimal** 循环小数 repeats the same digits forever. Dots mark the repeating part: $0.1\dot{7} = 0.1777\dots$ and $0.12\dot{3} = 0.123123\dots$

To turn a recurring decimal into a fraction, multiply so the repeating parts line up, then subtract.

Worked example. Write $0.1\dot{7}$ as a fraction.

Let $x = 0.1777\dots$. Only the 7 repeats, so use $10x$ and $100x$:

$$100x = 17.777\dots, \quad 10x = 1.777\dots$$

$$100x - 10x = 16, \quad 90x = 16, \quad x = \frac{16}{90} = \frac{8}{45}.$$

Write $x = 0.1\dot{7}$ as a fraction:

$100x = 17.$	$777\dots$	the repeating tails line up exactly
$10x = 1.$	$777\dots$	
$90x = 16$		
so $x = \frac{16}{90} = \frac{8}{45}$		

subtract: the tails **cancel** -
no dots left

Line up the repeating tails, subtract, and the tails cancel

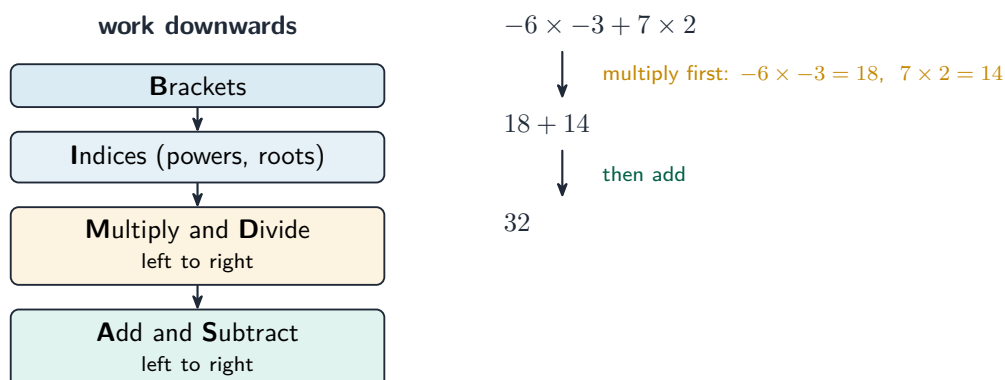
The four operations

Order of operations

Work in this order —the **order of operations** 运算顺序: brackets first, then indices (powers and roots), then multiply and divide (left to right), then add and subtract (left to right).

Worked example. Work out $-6 \times -3 + 7 \times 2$.

Do the multiplications first: $-6 \times -3 = 18$ and $7 \times 2 = 14$. Then add: $18 + 14 = 32$.



The order of operations, applied to the worked example

Negative numbers

- Adding a negative: $5 + (-3) = 5 - 3 = 2$.
- Subtracting a negative: $5 - (-3) = 5 + 3 = 8$.
- Multiplying or dividing: same signs give a positive; different signs give a negative. So $-6 \times -3 = 18$ but $-12 \div 4 = -3$.

A change in temperature from -5°C to 3°C is a rise of 8°C .

Calculating with fractions

- **Multiply:** multiply the tops, multiply the bottoms: $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.
- **Divide:** multiply by the reciprocal of the second fraction: $\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12} = \frac{5}{6}$.
- **Add or subtract:** use a common denominator (the LCM of the bottoms).

Worked example. Work out $1\frac{7}{15} - \frac{4}{5}$, giving the answer in its simplest form.

Change the mixed number to an improper fraction, then use denominator 15:

$$1\frac{7}{15} = \frac{22}{15}, \quad \frac{4}{5} = \frac{12}{15}, \quad \frac{22}{15} - \frac{12}{15} = \frac{10}{15} = \frac{2}{3}.$$

Ordering

Use these symbols to compare numbers by **magnitude** 大小 (size):

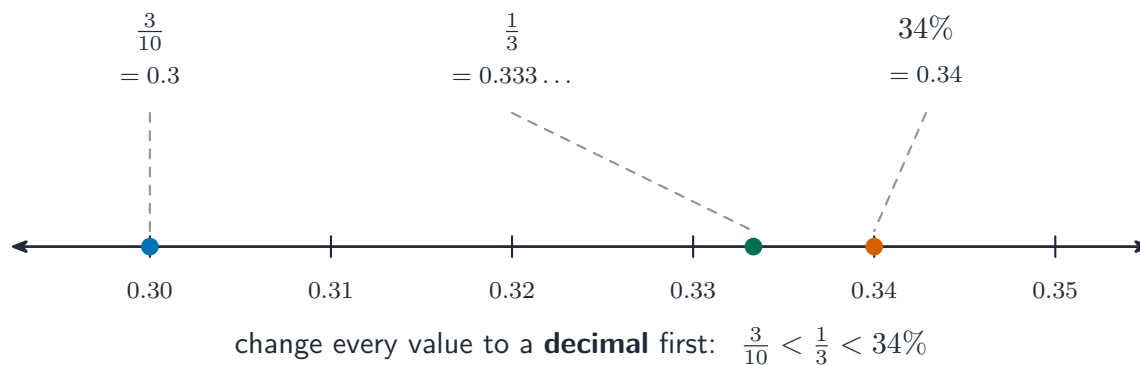
Symbol	Meaning
$=$	is equal to
$\neq$	is not equal to
$>$	is greater than
$<$	is less than
$\geq$	is greater than or equal to
$\leq$	is less than or equal to

To put a mixed list in order, change every value to a decimal first.

Worked example. Put 34%, $\frac{1}{3}$ and $\frac{3}{10}$ in order, smallest first.

As decimals: $34\% = 0.34$, $\frac{1}{3} = 0.333\dots$, $\frac{3}{10} = 0.3$. So the order is

$$\frac{3}{10} < \frac{1}{3} < 34\%.$$



Convert every value to a decimal, then place them on a number line

Percentages



Shops use percentages for discounts, sales tax and profit margins.

Image: John Baker, CC BY-SA 2.0 (commons.wikimedia.org)

Find a percentage of an amount. 15% of $\$80 = 0.15 \times 80 = \12 .

Write one amount as a percentage of another. A score of 18 out of 25 is $\frac{18}{25} \times 100\% = 72\%$.

Percentage increase or decrease uses

$$\text{percentage change} = \frac{\text{change}}{\text{original amount}} \times 100\%.$$

Worked example. A price rises from $\$40$ to $\$50$. Find the percentage increase.

The change is $\$10$, so $\frac{10}{40} \times 100\% = 25\%$ increase.

A quick way is a **multiplier** 乘数. To increase by 15%, multiply by 1.15; to decrease by 15%, multiply by 0.85.

Simple and compound interest

Interest 利息 is money paid for borrowing or for saving. The **principal** 本金 is the starting amount.

- **Simple interest** 单利 pays the same amount each year, worked out on the principal only:

$$I = \frac{P \times r \times t}{100},$$

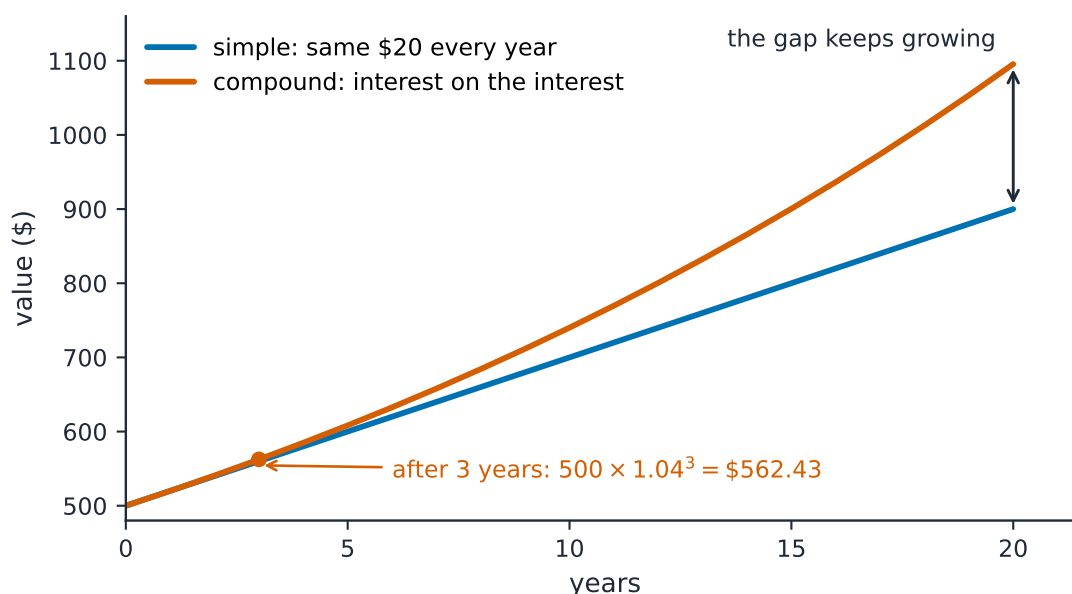
where P is the principal, r is the rate per year (as a percentage) and t is the number of years.

- **Compound interest** 复利 adds the interest on each year, so the next year earns interest on a larger total:

$$\text{final value} = P \left(1 + \frac{r}{100}\right)^t.$$

Worked example. Find the value of \$500 saved at 4% compound interest for 3 years.

$$500 \times 1.04^3 = 500 \times 1.124864 = \$562.43 \text{ (to the nearest cent).}$$



Simple interest grows in a straight line; compound interest grows faster every year

Reverse percentages (Extended)

A **reverse percentage** 逆百分比 problem gives the amount **after** a change and asks for the original. To solve it, divide by the multiplier —do not just take the percentage off.

Worked example. A coat costs \$60 after a 20% increase. Find the original price.

\$60 is 120% of the original, so the original price is $60 \div 1.2 = \$50$.

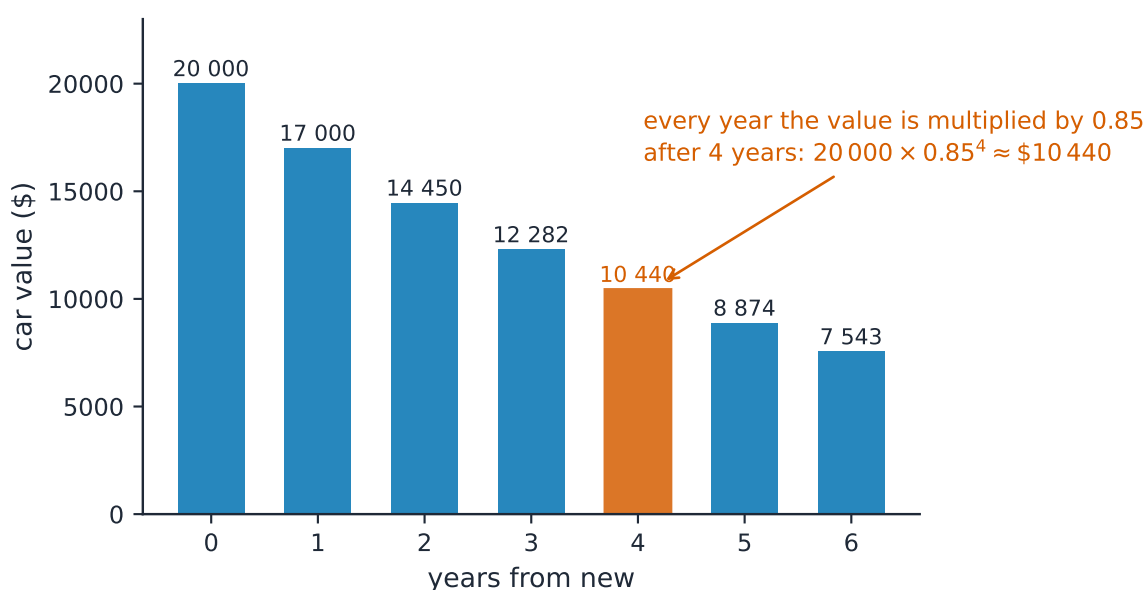
Exponential growth and decay (Extended)

When a quantity changes by the same percentage in each time period, it shows **exponential growth** 指数增长 (it gets bigger) or **exponential decay** 指数衰减 (it gets smaller). Use the compound formula. **Depreciation** 折旧, where something like a car loses value each year, is decay.

Worked example. A car worth \$20 000 loses 15% of its value each year. Find its value after 4 years.

The multiplier is 0.85, so

$$20\,000 \times 0.85^4 = 20\,000 \times 0.522 \dots = \$10\,440 \text{ (to the nearest dollar).}$$



Exponential decay: the car loses 15 percent of its current value every year

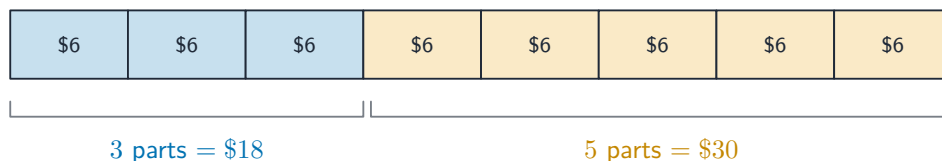
Ratio and proportion

A **ratio** 比 compares quantities, written like $a : b$. Simplify it like a fraction by dividing by the HCF: $20 : 30 : 40 = 2 : 3 : 4$.

Dividing in a ratio. Share \$48 in the ratio 3 : 5.

The total number of parts is $3 + 5 = 8$. One part is $48 \div 8 = \$6$. So the shares are $3 \times 6 = \$18$ and $5 \times 6 = \$30$.

Share \$48 in the ratio 3 : 5 - $3 + 5 = 8$ parts, one part = $48 \div 8 = \$6$



A bar model for sharing 48 dollars in the ratio 3 to 5

Proportion 比例 means two ratios are equal. Use it for recipes, map **scales** 比例尺 and finding best value.

Worked example. 3 pens cost \$1.80. Find the cost of 7 pens.

One pen costs $1.80 \div 3 = \$0.60$. So 7 pens cost $7 \times 0.60 = \$4.20$.

Rates and average speed

A **rate** 比率 compares two quantities measured in different units, such as price per kilogram, or distance per hour.

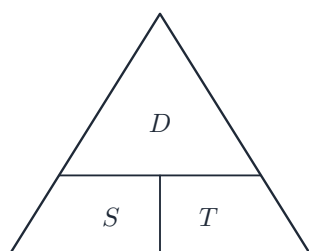
Average speed 平均速度 uses

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}.$$

Worked example. A cyclist travels 45 km in 3 hours 45 minutes. Find the average speed.

First change the time to hours: 3 h 45 min = 3.75 h. Then

$$\text{average speed} = \frac{45}{3.75} = 12 \text{ km/h}.$$



cover the one
you want

$$S = \frac{D}{T} \quad (\text{cover } S: D \text{ over } T)$$

$$D = S \times T \quad (\text{cover } D: S \text{ next to } T)$$

$$T = \frac{D}{S} \quad (\text{cover } T: D \text{ over } S)$$

$$\text{cyclist: } 45 \text{ km in } 3 \text{ h } 45 \text{ min} = 3.75 \text{ h} \Rightarrow S = \frac{45}{3.75} = 12 \text{ km/h}$$

The distance-speed-time triangle: cover the one you want

Other rates work the same way. **Density** 密度 is found from **mass** 质量 and **volume** 体积:

$$\text{density} = \frac{\text{mass}}{\text{volume}}.$$

Other examples are flow rate, fuel consumption and **population density** 人口密度. If a rate needs a special formula (such as **pressure** 压强), the question will give it to you.

Rounding and estimation

Rounding

- **decimal places (d.p.)** 小数位: digits counted after the point. 3.14159 to 2 d.p. is 3.14.
- **significant figures (s.f.)** 有效数字: digits counted from the first non-zero digit. 5764 to 1 s.f. is 6000; 0.004067 to 2 s.f. is 0.0041.

Rule: look at the next digit. If it is 5 or more, round up; if it is less, round down.

Estimation

To **estimate** 估算 an answer, round every number to 1 s.f., then calculate.

Worked example. Estimate $\frac{41.3}{9.79 \times 0.765}$.

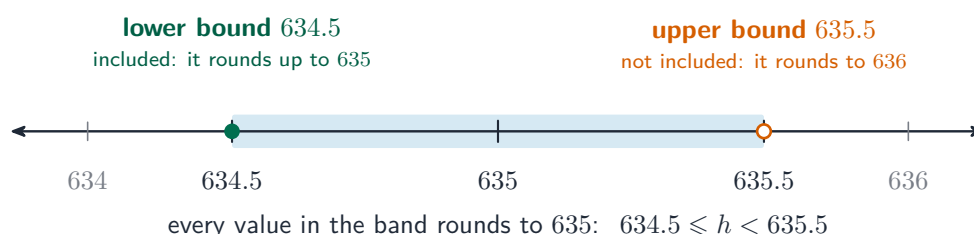
$$\frac{41.3}{9.79 \times 0.765} \approx \frac{40}{10 \times 0.8} = \frac{40}{8} = 5.$$

Limits of accuracy

A rounded value could really be anything that rounds to it. The smallest possible value is the **lower bound** 下界; the largest is the **upper bound** 上界. For a value rounded to the nearest unit, the bounds lie half a unit on each side.

Worked example. A height h is 635 m, correct to the nearest metre. Give the bounds.

$$634.5 \leq h < 635.5.$$



Everything in the band rounds to 635: the lower bound is included, the upper bound is not

So the lower bound is 634.5 m and the upper bound is 635.5 m.

Bounds in calculations (Extended)

Combine the bounds to get the bound you want.

Worked example. A rectangle is 8 cm by 5 cm, each side to the nearest cm. Find the largest possible **area** 面积.

Use the upper bounds of both sides: $8.5 \times 5.5 = 46.75 \text{ cm}^2$. (The smallest area uses the lower bounds: $7.5 \times 4.5 = 33.75 \text{ cm}^2$.)

For a divided quantity such as $\text{speed} = \frac{\text{distance}}{\text{time}}$, the **largest** speed comes from the **largest** distance divided by the **smallest** time.

Time

- 60 seconds = 1 minute, 60 minutes = 1 hour, 24 hours = 1 day, and 1 year = 365 days.
- The 24-hour clock writes a time as four digits: 3.15 p.m. is 15 15.

Worked example. A film starts at 19 35 and lasts 70 minutes. Find the time it finishes.

70 min = 1 h 10 min. Adding 1 hour gives 20 35; adding 10 minutes gives 20 45.

For timetables and **time zone** 时区 problems, add or subtract the time difference between the places.

Money

Work with money as ordinary decimals, but give answers to 2 d.p. (so \$4.8 is written \$4.80).

Currency conversion. Use the **exchange rate** 汇率 as a multiplier.

Worked example. The exchange rate is $\$1 = \text{€}0.92$. Convert \$150 to euros, and convert €138 back to dollars.

$$150 \times 0.92 = \text{€}138, \quad 138 \div 0.92 = \$150.$$

Using a calculator

- Do not round part-way through a calculation. Keep the full value and round only the final answer.
- Enter time as a decimal of an hour: 2 hours 30 minutes is 2.5 hours, not 2.30.
- Read the display in context: in money, 4.8 means \$4.80; in time, 3.25 hours means 3 hours 15 minutes.

Exam tips

- Follow **BIDMAS** (brackets, indices, division/multiplication, addition/subtraction) in order, and remember a negative times a negative is positive.
- In **standard form** the number in front is between 1 and 10; a small number (like 0.0004) has a *negative* power of 10.
- A percentage *change* is worked out on the **original** amount. For a reverse percentage, divide by the multiplier (e.g. $\div 1.2$ undoes a 20% rise).
- Do not round part-way through —keep the full value and round only at the end, to the accuracy the question asks for (decimal places or significant figures).
- For **limits of accuracy**, a value rounded to the nearest whole number can be up to 0.5 either side (so 8 means $7.5 \leq x < 8.5$).