Motion in a circle

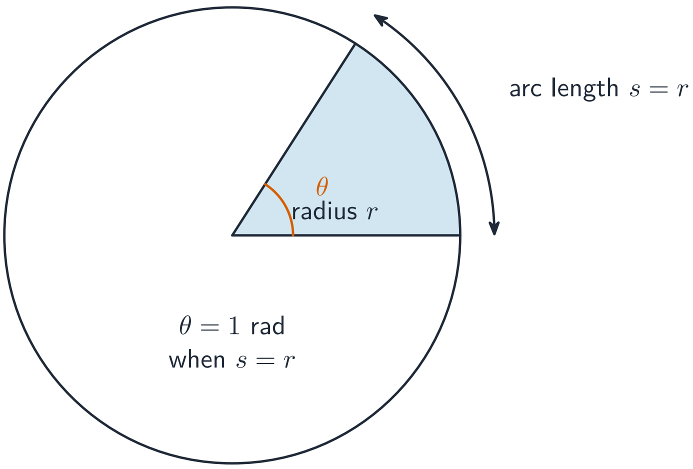
A-Level Physics

Angles in radians

The **radian** 弧度 is the angle made at the centre of a circle by an **arc** 弧 whose length equals the radius. For an arc of length s on a circle of radius r, the angle in radians is

$$\theta = \frac{s}{r}.$$

Radians have no unit (a ratio of lengths). A full circle has $s = 2\pi r$, so $\theta = 2\pi$ rad. A half-circle is π rad; a quarter is $\pi/2$ rad.



One radian is the angle whose arc length equals the radius

To convert: 1 rad = $180°/\pi \approx 57.3°$. Set your calculator to **radians** for this topic; ”degree” mode will give wrong answers.

Uniform circular motion: angular speed

A spinning fairground ride: every rider turns through the same angle each second.

An object moves in a circle of radius r at constant speed v. Define:

- **angular displacement** 角位移 θ —the angle (in radians) turned through by the radius from a chosen start line.
- **angular speed** 角速度 ω —the rate of change of angular displacement.

For uniform motion ω is constant and

$$\omega = \frac{\theta}{t}.$$

Unit: rad s^{-1}.

Period and frequency

If the object goes once round (2π rad, one **revolution** 圈) in time T (the **period** 周期), then

$$\omega = \frac{2\pi}{T} = 2\pi f,$$

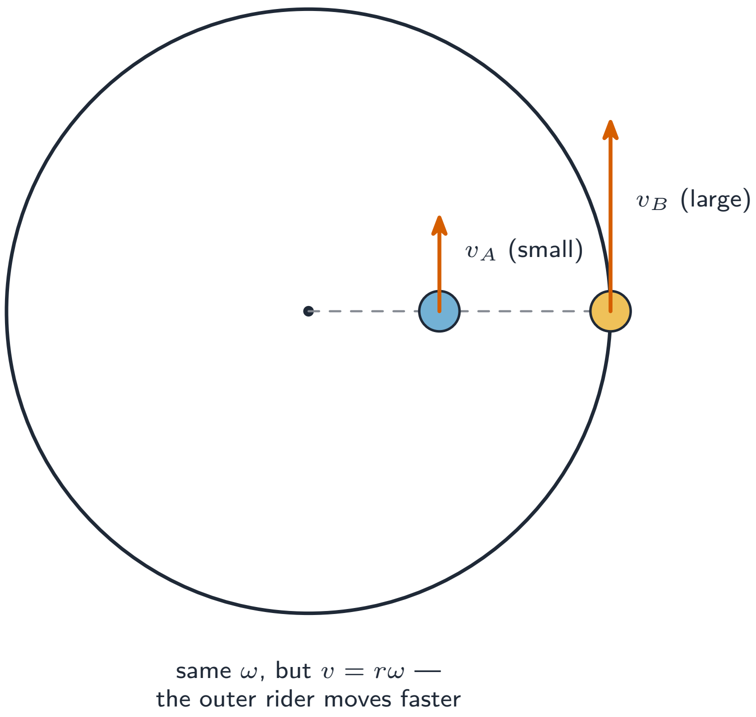
where $f = 1/T$ is the **frequency** 频率 of turning (Hz).

Linear and angular speed

In one period T the object travels a distance $2\pi r$ (the **circumference** 周长) at constant speed, so

$$v = \frac{2\pi r}{T} = r\omega.$$

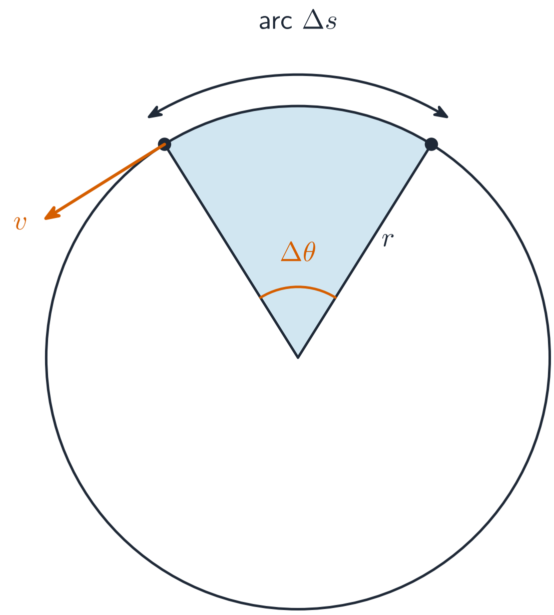
This links the linear (**tangential 切向**) speed v with the angular speed ω. At a larger radius (for the same angular speed) the linear speed is larger —a child on the edge of a merry-go-round moves faster than one near the centre, even though both go round once in the same time.



Same angular speed, but the rider at the larger radius has the larger linear speed ($v = r\omega$)

Worked example. A fairground ride of radius 4.0 m completes one turn every 8.0 s. Find its angular speed and the linear speed of a rider on the edge.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8.0} = 0.79 \text{ rad s}^{-1}, \qquad v = r\omega = 4.0 \times 0.79 = 3.1 \text{ m s}^{-1}.$$



As the radius turns through $\Delta\theta$ the object moves an arc Δs at speed v

Centripetal acceleration

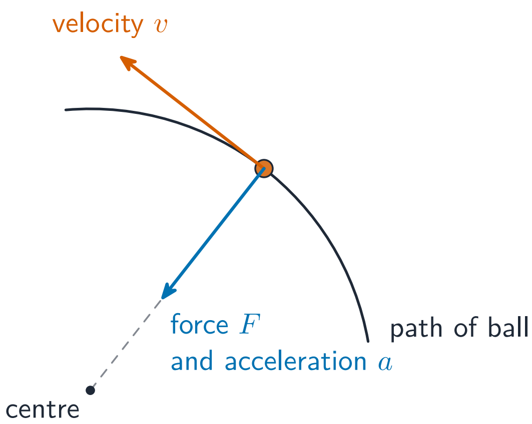
An object moving in a circle at constant speed still has a **changing velocity** 速度—its direction keeps changing, even though its size stays the same. A changing velocity needs an **acceleration** 加速度. This acceleration points **towards the centre** and is the **centripetal acceleration** 向心加速度.

Size

$$a = \frac{v^2}{r} = r\omega^2.$$

The two forms are equal because $v = r\omega$. Pick the one with the quantities you have.

The centripetal acceleration is **perpendicular** 垂直 to the velocity at every instant — never along the direction of motion. (If part of it were along the motion, the speed would change.) Unit: m s^{-2}.



The velocity points along the tangent; the force and acceleration point to the centre

Centripetal force

A Ferris wheel: a centripetal force toward the centre keeps each car moving in a circle.

By Newton's second law, the resultant **force 力** on a body in circular motion at constant speed is

$$F = ma = \frac{mv^2}{r} = mr\omega^2.$$

This is the **centripetal force 向心力**. It **always points towards the centre** — perpendicular to the velocity.

The centripetal force is not a new kind of force —it is the **net result** of the real forces acting (tension, gravity, friction, electric attraction, normal contact force, $\cdots$). In a problem, work out which real force(s) provide it.

Worked example. A 0.20 kg ball on a string is whirled in a horizontal circle of radius 0.50 m at 3.0 m s^{-1}. Find the centripetal force (the tension in the string).

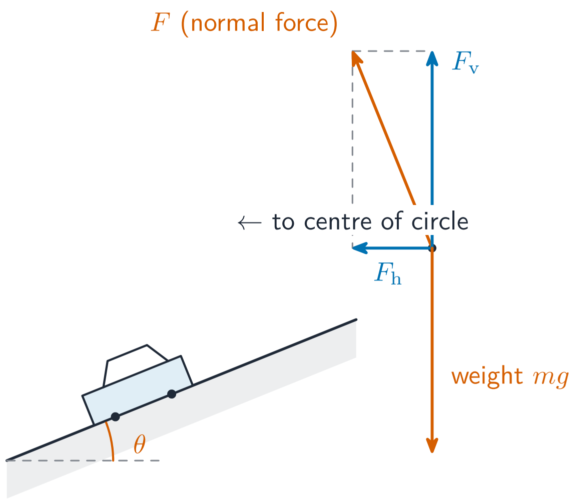
$$F = \frac{mv^2}{r} = \frac{0.20 \times 3.0^2}{0.50} = 3.6 \text{ N}.$$

Where the centripetal force comes from

- **Ball on a string** in a horizontal circle: the **tension** 张力 in the string.
- **Car turning a flat corner**: the **friction** 摩擦力 between tyres and road ($F = mv^2/r$). If the car goes too fast, friction is not enough and it skids outwards.
- **Banked corner** 倾斜 (no friction): the horizontal part of the **normal contact force** 支持力; $\tan\theta = v^2/(rg)$ for the angle that needs no friction.
- **Planet or satellite** 卫星 in **orbit** 轨道: the **gravitational attraction** 引力, $GMm/r^2 = mv^2/r$.
- **Electron** 电子 in a circular orbit (Bohr-style model): the **electrostatic** 静电 attraction between the electron and the positive **nucleus** 原子核:

$$\frac{kZe^2}{r^2} = \frac{m_e v^2}{r},$$

where $k = 1/(4\pi\varepsilon_0)$ and Z is the nuclear charge. Solve for v to get the orbital speed; then $T = 2\pi r/v$.

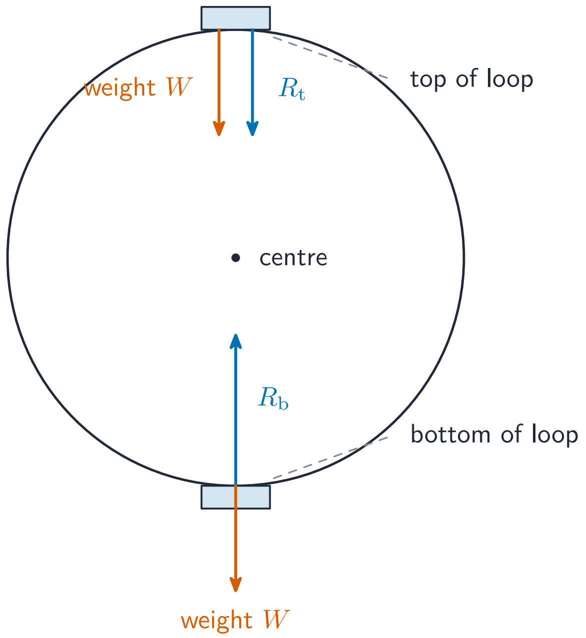


On a banked track the horizontal part of the road's force provides the centripetal force

Vertical circles

When the circle is upright, the speed is **not** constant (gravity does work) —but at each instant the **net force towards the centre** still equals mv^2/r:

- at the bottom of a loop: tension up, **weight 重力** down, so $T - mg = mv^2/r$ —the tension is largest here.
- at the top of a loop: tension and weight both point down (towards the centre), so $T + mg = mv^2/r$ —the tension is smallest. For the slowest speed at the top with the string just tight, set $T = 0$: $mg = mv_{\text{min}}^2/r$, giving $v_{\text{min}} = \sqrt{gr}$.



Forces on a person at the top and bottom of a vertical circle

Worked example. A car goes round a vertical loop of radius 2.0 m. Find the minimum speed at the top for the car to keep contact with the track (take $g = 9.81$ m s^{-2}).

At the slowest speed the track force is zero, so gravity alone provides the centripetal force: $mg = mv_{\text{min}}^2/r$, giving $v_{\text{min}} = \sqrt{gr}$:

$$v_{\text{min}} = \sqrt{9.81 \times 2.0} \approx 4.4 \text{ m s}^{-1}.$$

The constant-speed result ($v = r\omega$, ω constant) holds for horizontal circles, or where the force only bends the path (orbits in gravity, charges in a **magnetic field 磁场**).

How to structure a circular-motion answer

1. **Find the radius** r and choose v or ω. Use $v = r\omega$ to switch between them.
2. **Find the centripetal acceleration** with $a = v^2/r$ or $r\omega^2$.
3. **List the real forces** and write Newton's second law in the **radial 径向** direction (towards the centre is positive). Set the net inward force equal to mv^2/r.
4. For **period or frequency**: use $\omega = 2\pi/T$, or $T = 2\pi r/v$.
5. Check the **directions**: centripetal force and acceleration point to the centre; the velocity is along the tangent.

Exam tips

- Work in **radians**; angular speed $\omega = 2\pi/T = v/r$.
- **Centripetal acceleration** $a = v^2/r = \omega^2 r$; the net force acts **towards the centre** —it is provided by tension/gravity/friction, not an extra force.
- Always state **what provides the centripetal force** in the situation given.