Deformation of solids

A-Level Physics

Forces that cause deformation

When a force acts on a solid along its length, the object changes shape (**deformation** 形变). Two cases (treated as one-dimensional here):

- a **tensile** 拉伸 force stretches the object —it makes an **extension** 伸长量 x,
- a **compressive** force squeezes the object —it makes a **compression** 压缩, treated as a negative extension.

The applied force is the **load** 负载. The change from the natural length is the extension (or compression).

Hooke's law and the spring constant

A spring obeys Hooke's law: extension is proportional to the force applied.

Image: Simon Speed, Public domain (commons.wikimedia.org)

For many materials at small extensions, **the extension is proportional to the load** —this is **Hooke's law** 胡克定律. The constant that links them is the **spring constant** 劲度系数 k:

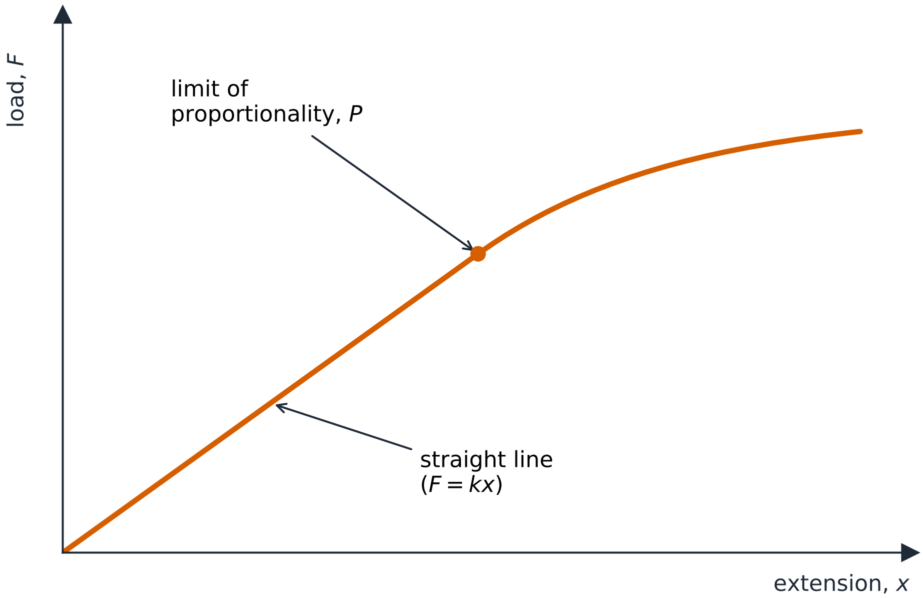
$$F = kx \qquad \Longleftrightarrow \qquad k = \frac{F}{x}.$$

Unit of k: N m^{-1}.

Worked example. A spring stretches by 4.0 cm when a 2.0 N load is hung from it. Find its spring constant.

Converting the extension to metres (4.0 cm = 0.040 m):

$$k = \frac{F}{x} = \frac{2.0}{0.040} = 50 \text{ N m}^{-1}.$$



A load–extension graph: straight up to the limit of proportionality P, then it curves

Reading a graph:

- A force–extension (F against x) graph has gradient k in the Hooke's-law region.
- An extension–force (x against F) graph has gradient $1/k$ in the Hooke's-law region.

A common trap: if a graph plots **length** L against force, you can still find the spring constant from the gradient (since $L = L_0 + F/k$, the gradient is $1/k$ —read it off carefully).

Limit of proportionality

Hooke's law only holds up to the **limit of proportionality** 比例极限. Past this point the F against x line curves and is no longer straight. The material may still be **elastic** 弹性 (it returns to its first length when you remove the load) a little further, then it becomes **plastic** 塑性.

Springs in series and parallel

You may need to combine spring constants:

- **Series** 串联 (one spring hangs from another): the same load passes through both, the total extension is the sum, so $\frac{1}{k_{\text{total}}} = \frac{1}{k_1} + \frac{1}{k_2}$.

- **Parallel** 并联 (two springs side by side holding the same load): each takes half the load (if they are identical), the extensions are equal, so $k_{\text{total}} = k_1 + k_2$.

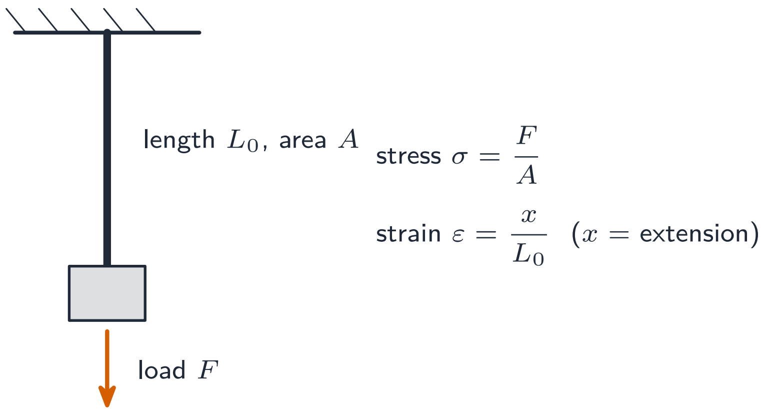
$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \qquad\qquad k = k_1 + k_2$$

Combining spring constants: series gives a softer spring, parallel a stiffer one

Stress, strain and the Young modulus

For a wire of uniform cross-section under a tensile load:

- **Stress** 应力 $\sigma = \dfrac{F}{A}$, where F is the load and A is the **cross-sectional area** 横截面积. Unit: Pa.
- **Strain** 应变 $\varepsilon = \dfrac{x}{L_0}$, where x is the extension and L_0 is the original length. Strain has no unit (it is a ratio of lengths).



Stress is the load per cross-sectional area; strain is the extension per original length

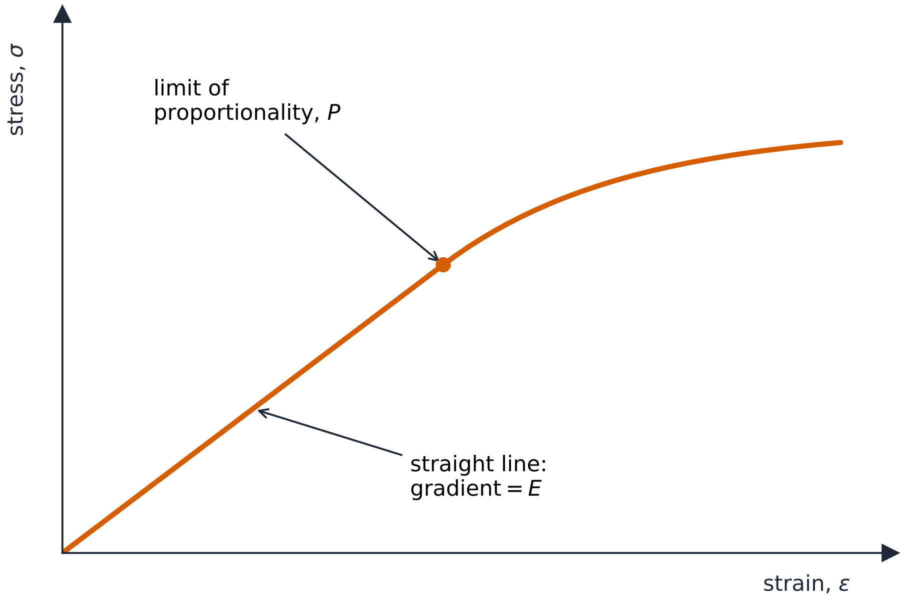
The **Young modulus** 杨氏模量 is the ratio of stress to strain in the Hooke's-law region:

$$E = \frac{\sigma}{\varepsilon} = \frac{F/A}{x/L_0} = \frac{FL_0}{Ax}.$$

Unit: Pa (about 10^{11} for metals; e.g. steel $\approx 2.0 \times 10^{11}$ Pa).

Worked example. A steel wire of length 2.0 m and cross-sectional area 1.5×10^{-7} m^2 stretches by 1.0 mm under a load of 15 N. Find the Young modulus.

$$E = \frac{FL_0}{Ax} = \frac{15 \times 2.0}{(1.5 \times 10^{-7})(1.0 \times 10^{-3})} = 2.0 \times 10^{11} \text{ Pa}.$$



A stress–strain graph, straight up to the limit of proportionality P

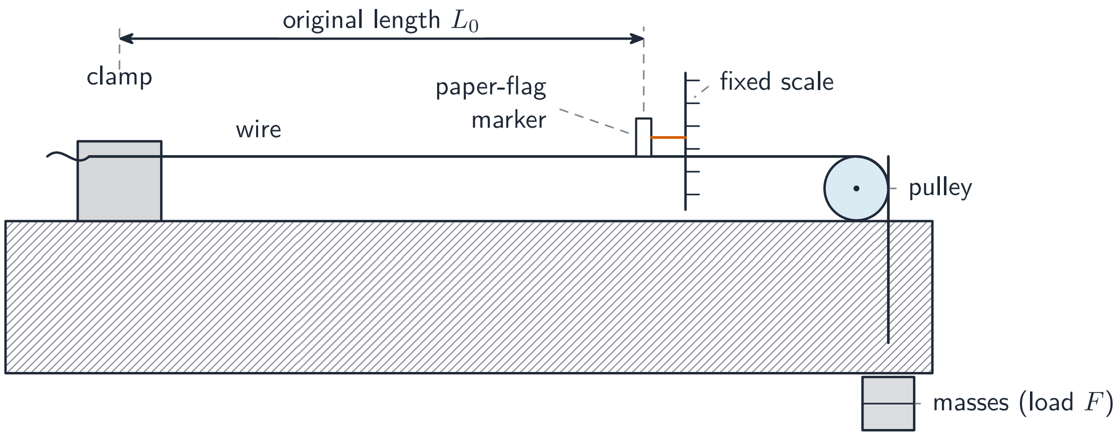
The Young modulus is a property of the **material** —it does not depend on the wire's shape or size. The spring constant k depends on both the material and the size: $k = EA/L_0$.

Experiment to find the Young modulus of a metal wire

A standard setup:

1. Clamp one end of a long, thin wire to a fixed support. Pass the wire over a **pulley** 滑轮 at the edge of the bench so it hangs straight down.
2. Measure the original length L_0 between the clamp and a marker near the pulley, using a metre rule.
3. Measure the **diameter** 直径 d of the wire at several places with a **micrometer** 螺旋测微器 and take the average. Work out $A = \pi d^2/4$.
4. Hang weights one at a time. Record the load F and the extension x (how far the marker moves against a fixed scale).
5. Plot F against x. In the straight region the gradient is EA/L_0, so $E = \text{gradient} \times L_0/A$.

Why a **long, thin** wire? To make the extension big enough to measure well. Why repeat readings and measure d at several places? To reduce **random error** 随机误差 and check the wire is uniform.



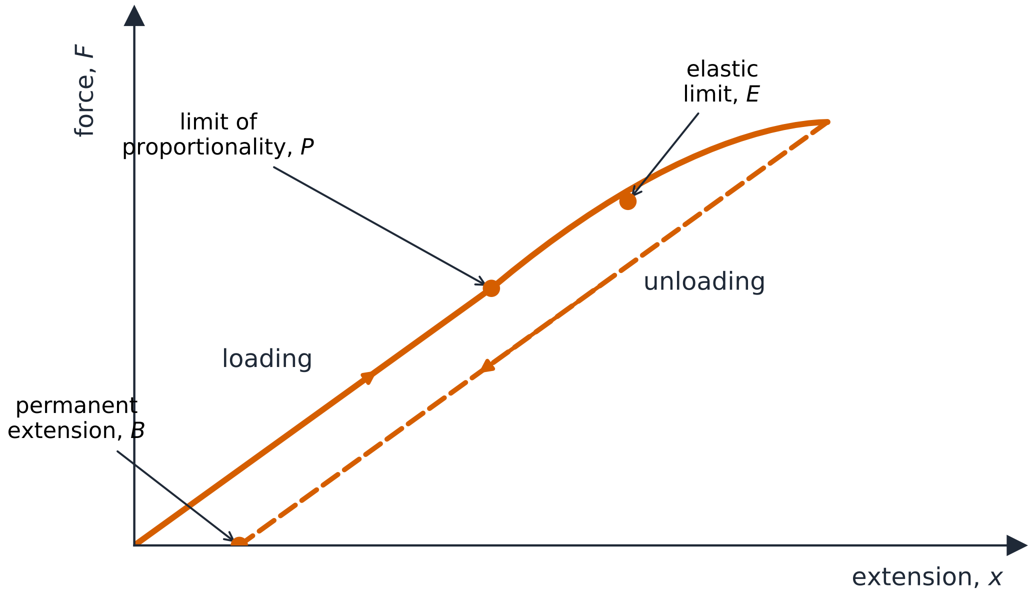
Apparatus for measuring the Young modulus of a wire

Elastic and plastic behaviour

As the load grows:

1. **Elastic and straight (Hooke obeyed)** —up to the limit of proportionality. Removing the load returns the object to its first length.
2. **Elastic but curved** —between the limit of proportionality and the **elastic limit** 弹性极限. The extension is no longer straight in the load, but on unloading the object still returns to its first length.
3. **Plastic** —past the elastic limit. On unloading, the object does **not** return to its first length; a permanent extension stays.

Hooke's law only holds in the straight, elastic region.



Force–extension past the elastic limit: P and E marked, with a permanent extension B left after unloading

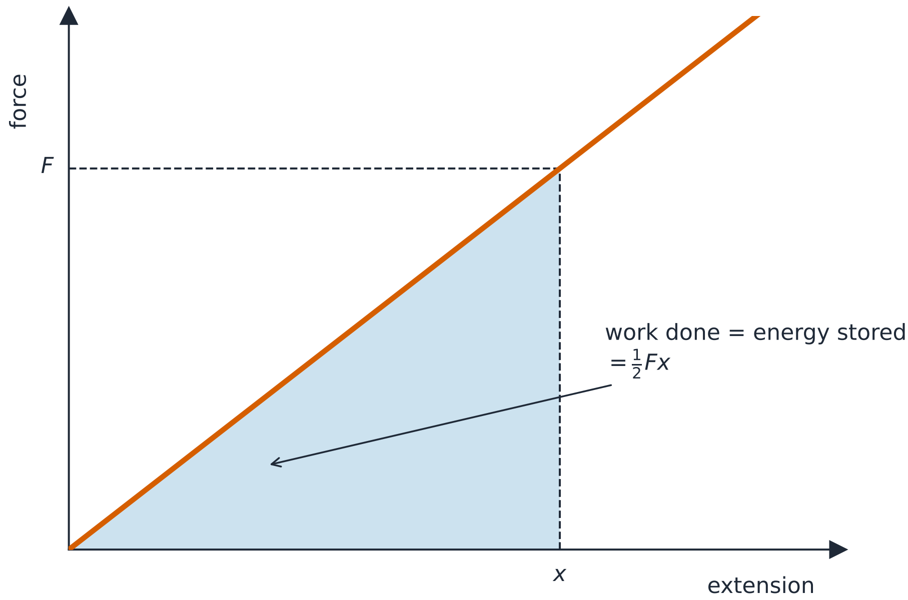
A modern universal (tensile) testing machine stretches a sample and records the force and extension

Image: Shimadzu, Product image (shimadzu-testing.com)

On a force–extension graph for a material taken into the plastic region and then unloaded, the loading line and the unloading line are different. The unloading line is parallel to the first Hooke line but shifted to the right (the permanent extension left when the load reaches zero). The area between the loading and unloading lines is the energy turned into **thermal energy** 热能 in the material.

Energy stored in a stretched material

The **work done** in stretching a material from 0 to extension x, as the load grows from 0 to F, is the **area under the force–extension graph**.



The work done stretching a material is the area under the force–extension graph

Hooke's-law material

When Hooke's law holds, the F against x graph is a straight line through the origin. The area under it from 0 to x is a triangle:

$$E_{\mathrm{P}} = \tfrac{1}{2}Fx = \tfrac{1}{2}kx^2.$$

This is the **elastic potential energy 弹性势能** stored in a spring or wire stretched within its limit of proportionality. An equal form:

$$E_{\mathrm{P}} = \frac{F^2}{2k}.$$

Worked example. A spring of spring constant 50 N m^{-1} is stretched by 0.20 m, within its limit of proportionality. Find the elastic potential energy stored.

$$E_{\mathrm{P}} = \tfrac{1}{2}kx^2 = \tfrac{1}{2} \times 50 \times 0.20^2 = 1.0 \text{ J}.$$

Non-Hooke material

For a graph that is not a straight line (a stretched rubber band, or a spring past its limit of proportionality), find the area by counting grid squares or by using **trapezia 梯形**. The same idea holds: **the area under the force–extension graph is the work done on the material**.

Comparing stored energy

A common multiple-choice case: two materials are stretched by the same force, or by the same extension. Using $E_P = \frac{1}{2}Fx$:

- same F, smaller k (less stiff) $\rightarrow$ larger x $\rightarrow$ more energy stored.
- same x, larger k (stiffer) $\rightarrow$ larger F $\rightarrow$ more energy stored.

When a stretched spring is released onto a mass, the elastic potential energy becomes **kinetic energy 动能** (and gravitational potential energy if the mass rises). Set $\frac{1}{2}kx^2$ equal to $\frac{1}{2}mv^2$ (plus any mgh) to find the speed or height.

Exam tips

- **Hooke's law (**$F = kx$**)** holds only up to the limit of proportionality.
- **Stress** $= F/A$**, strain** $= x/L$**, Young modulus** $=$ **stress/strain** (gradient of the straight part of the stress-strain graph) —watch the units (Pa).
- Energy stored $=$ **area under the force-extension graph** $= \frac{1}{2}Fx$ in the elastic region.
- Distinguish **elastic** (returns to shape) from **plastic** (permanent) deformation.