Pure Mathematics 3

A-Level Mathematics

This handout covers Topic 3: **Pure Mathematics** 纯数学 3. Subtopics 3.1–3.6 build on the algebra, logarithms, trigonometry, differentiation, integration and numerical methods of Pure Mathematics 2, so this handout explains what is **new** in Pure 3 and then covers the three big new areas: vectors, differential equations and complex numbers.

Algebra

Two new tools join the algebra from Pure 2.

Partial fractions

A single fraction with a factorised bottom can be split into a sum of simpler fractions. This is called writing it in **partial fractions** 部分分式, and it makes a **rational function** 有理函数 (a fraction of polynomials) easy to integrate or expand. Match the form to the bottom:

First check the top is **lower degree** than the bottom. If it is not ("top-heavy"), **divide** the polynomial first: dividing gives a **quotient** 商 plus a **remainder** 余数 over the original bottom, and you split only that remaining proper fraction.

$$\frac{x+4}{(x+1)(x-2)} \xrightarrow{\text{split}} \frac{2}{x-2} - \frac{1}{x+1}$$

One fraction with a factorised bottom splits into a sum of simpler fractions.

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}, \qquad \frac{1}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}.$$

Worked example. Express $\dfrac{x+4}{(x+1)(x-2)}$ in partial fractions.

Write $\dfrac{x+4}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$, so $x+4 = A(x-2) + B(x+1)$. Put $x = 2$: $6 = 3B$, so $B = 2$. Put $x = -1$: $3 = -3A$, so $A = -1$. Hence

$$\frac{x+4}{(x+1)(x-2)} = \frac{2}{x-2} - \frac{1}{x+1}.$$

The binomial expansion for a rational power

The **binomial expansion** 二项展开式 also works when the power is a fraction or is negative, as long as $|x| < 1$:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

For example $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$ for $|x| < 1$.

Logarithms, trigonometry and numerical methods (from Pure 2)

Subtopics 3.2, 3.3 and 3.6 are the same skills you met in Pure 2: the laws of logarithms with e^x and $\ln x$; the identities $\sec^2\theta \equiv 1+\tan^2\theta$ and $\csc^2\theta \equiv 1+\cot^2\theta$, the compound- and double-angle formulae, and the R-form of $a\sin\theta + b\cos\theta$; and solving an equation numerically by a **sign change** 变号 and an **iterative formula** 迭代公式 $x_{n+1} = F(x_n)$. Use them exactly as before.

The three new trig functions are the **reciprocals** 倒数 of the familiar ones: the **secant** 正割 $\sec\theta = \dfrac{1}{\cos\theta}$, the **cosecant** 余割 $\csc\theta = \dfrac{1}{\sin\theta}$, and the **cotangent** 余切 $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$. (Memory aid: match the *third* letter —se**c** goes with **c**osine.) The two Pythagorean identities above come straight from dividing $\sin^2\theta + \cos^2\theta \equiv 1$ by $\cos^2\theta$ or $\sin^2\theta$. For a numerical method, an iteration $x_{n+1} = F(x_n)$ **converges** 收敛 to the root when successive values get closer together.

Differentiation

The methods are those of Pure 2 (the product, quotient and chain rules, with parametric and implicit curves). One derivative is added —the **inverse tangent** 反正切:

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}.$$

Worked example. Differentiate $y = \tan^{-1}(3x)$.

Use the chain rule with the result above: $\dfrac{dy}{dx} = \dfrac{1}{1+(3x)^2} \times 3 = \dfrac{3}{1+9x^2}$.

Integration

Pure 3 adds several powerful methods.

- A new standard integral: $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$.
- **Partial fractions**: split a rational function first, then integrate each piece as a logarithm.
- The pattern $\dfrac{k\,f'(x)}{f(x)}$: this integrates to $k\ln|f(x)|+C$. For example $\displaystyle\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1)+C$.
- **Integration by parts 分部积分**, used for a product: $\displaystyle\int u\,\frac{dv}{dx}\,dx = uv - \int v\,\frac{du}{dx}\,dx$.
- **Integration by substitution 换元积分**: a given change of variable turns a hard integral into an easy one.

Worked example. Find $\displaystyle\int x\cos x\,dx$.

Use integration by parts with $u = x$ and $\dfrac{dv}{dx} = \cos x$, so $\dfrac{du}{dx} = 1$ and $v = \sin x$:

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Vectors

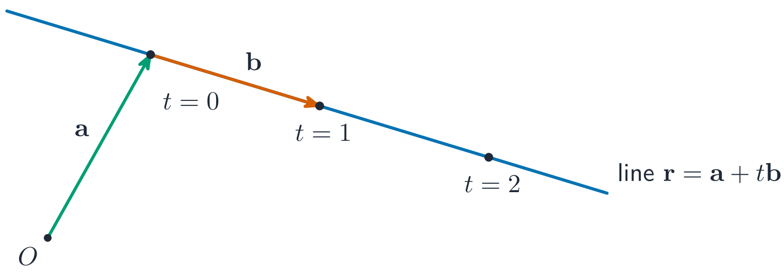
Forces like wind and water are vectors —they have both size and direction.

A **vector** 向量 has both size and direction. Write it as a column, or as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, or as $\overrightarrow{AB}$.

- The **magnitude** 模长 (length) of $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$.
- A **unit vector** 单位向量 has magnitude 1; divide a vector by its magnitude to make one.
- A **position vector** 位置向量 gives a point's place from the origin; a **displacement vector** 位移向量 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ goes from one point to another. Multiplying by a **scalar** 标量 (a plain number) stretches a vector.

Lines and the scalar product

A straight line through point $\mathbf{a}$ in direction $\mathbf{b}$ has vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. Two lines may be **parallel** 平行, may **intersect** 相交 at a point, or may be **skew lines** 异面直线 (not parallel and never meeting).

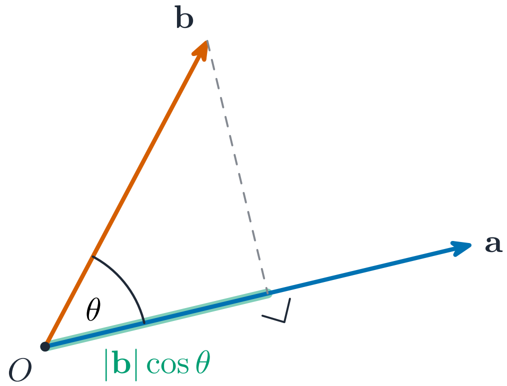


Start at the point $\mathbf{a}$, *then add* t *copies of the direction* $\mathbf{b}$ *to reach any point on the line.*

The **scalar product** 数量积 (dot product) of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}|\,|\mathbf{b}| \cos\theta,$$

where θ is the angle between them. So $\mathbf{a} \cdot \mathbf{b} = 0$ means the vectors are perpendicular.



The scalar product picks out $|\mathbf{b}|\cos\theta$*, how far* $\mathbf{b}$ *reaches along* $\mathbf{a}$.

Worked example. Find the angle between $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

$$\mathbf{a}\cdot\mathbf{b} = (1)(2) + (2)(2) + (2)(1) = 8, \qquad |\mathbf{a}| = |\mathbf{b}| = 3.$$

So $\cos\theta = \dfrac{8}{3\times 3} = \dfrac{8}{9}$, giving $\theta = 27.3°$.

Differential equations

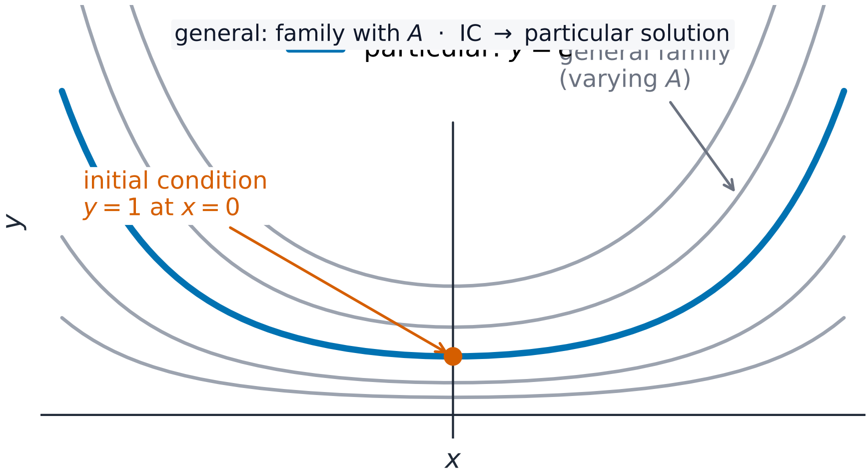
A **differential equation** 微分方程 links a quantity to its **rate of change**. To solve a first-order equation whose variables are **separable** 可分离变量, put all the y terms on one side and all the x terms on the other, then integrate both sides. This gives the **general solution** 通解, which contains a constant. An **initial condition** 初始条件 (a known value) fixes the constant and gives the **particular solution** 特解.

Worked example. Solve $\dfrac{dy}{dx} = xy$, given that $y = 1$ when $x = 0$.

Separate the variables and integrate:

$$\int \frac{1}{y}\,dy = \int x\,dx \;\Rightarrow\; \ln y = \tfrac{1}{2}x^2 + c \;\Rightarrow\; y = Ae^{x^2/2}.$$

Using $y = 1$ at $x = 0$ gives $A = 1$, so $y = e^{x^2/2}$.



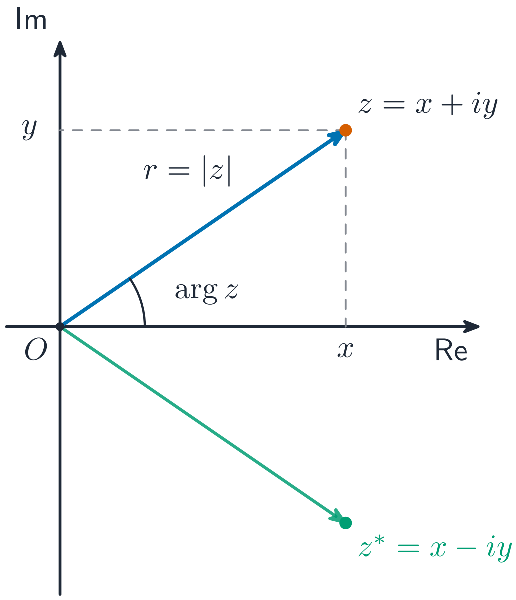
The constant A gives a whole family of curves; the condition $y = 1$ at $x = 0$ selects $y = e^{x^2/2}$.

Complex numbers

Self-similar patterns like Romanesco arise from iterating functions in the complex plane.

A **complex number** 复数 has the form $z = x + iy$, where $i^2 = -1$. Here x is the **real part** 实部 and y is the **imaginary part** 虚部. This $x + iy$ is the **Cartesian form** 直角坐标形式. Two complex numbers are equal only when their real parts match and their imaginary parts match.

- The **conjugate** 共轭 of $z = x + iy$ is $z^* = x - iy$. For a polynomial with real coefficients, any non-real roots come in conjugate pairs.
- The **modulus** 模 is $|z| = \sqrt{x^2 + y^2}$ (its distance from the origin) and the **argument** 辐角 is the angle the point makes, measured from the positive real axis.
- You can plot z as a point on an **Argand diagram** 阿干图 (the complex plane).
- The **polar form** 极坐标形式 is $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where $r = |z|$ and θ is the argument. Multiplying multiplies the moduli and adds the arguments.
- The **loci** of points satisfying a condition on z are drawn on the Argand diagram—e.g. $|z - a| = r$ is a circle, $|z - a| = |z - b|$ a perpendicular bisector, and $\arg(z - a) = \theta$ a half-line. The **square roots** of a complex number come from solving $w^2 = z$.



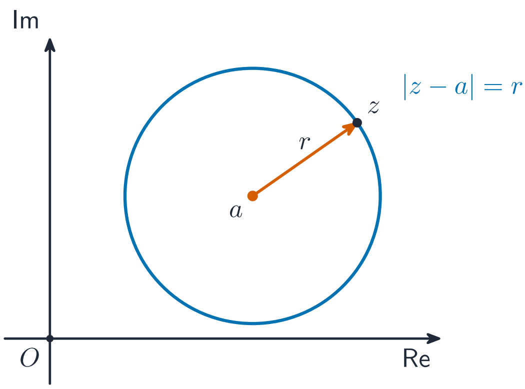
On the Argand diagram $|z|$ is the distance from O, $\arg z$ the angle, and z^* the reflection in the real axis.

To divide, multiply top and bottom by the conjugate of the bottom.

Worked example. Write $\dfrac{3+i}{1-i}$ in the form $x + iy$.

$$\frac{3+i}{1-i} = \frac{(3+i)(1+i)}{(1-i)(1+i)} = \frac{3+3i+i+i^2}{1+1} = \frac{2+4i}{2} = 1+2i.$$

An equation or inequality in z describes a **locus** 轨迹 (a path or region) on the Argand diagram. For example $|z-a| = r$ is a circle of radius r centred at a.



$|z-a| = r$ is the set of points a fixed distance r from a —a circle.

Exam tips

- Split a rational function into **partial fractions** before integrating or expanding.
- Choose the right **integration technique** (substitution, by parts, or partial fractions) from the form of the integrand.
- Use the scalar (dot) product for the **angle between vectors** and to test for perpendicularity; write a line as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.
- Give **complex numbers** in the form asked for (Cartesian or modulus-argument) and show them on an Argand diagram.