

Pure Mathematics 1

A-Level Mathematics

This handout covers Topic 1: **Pure Mathematics** 纯数学 1. It is the algebra 代数 and calculus 微积分 core of the course. Each ## section is one syllabus subtopic.

Quadratics



A suspension bridge: the main cable hangs in a parabola.

A **quadratic** 二次式 is an expression of the form $ax^2 + bx + c$, where $a \neq 0$. The letters a , b , c are the **coefficients** 系数 (the fixed numbers). Much of this section is about solving the **equation** $ax^2 + bx + c = 0$.

Completing the square

To **complete the square** 配方 means to write the quadratic in the form

$$a(x + p)^2 + q.$$

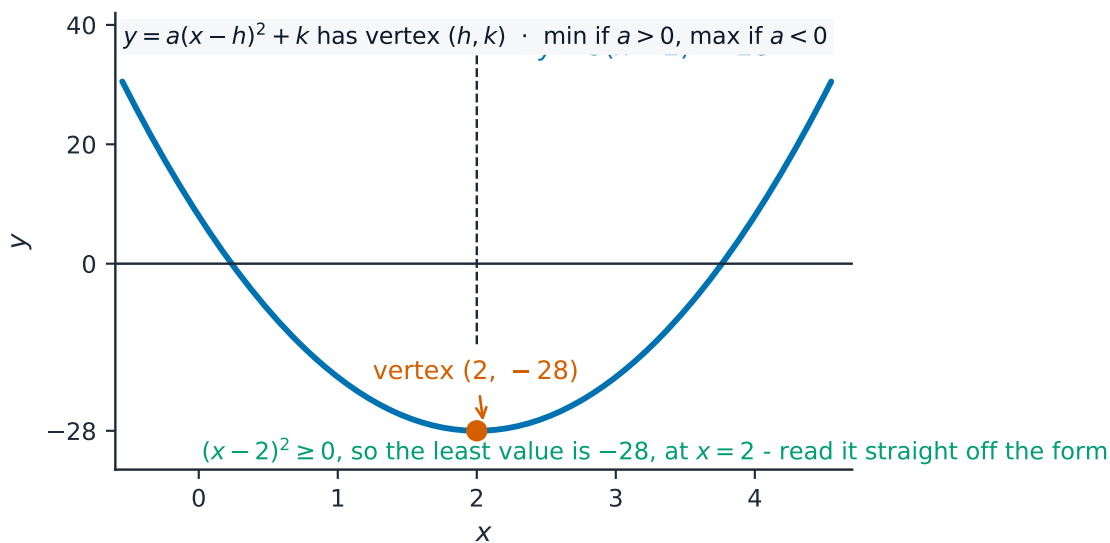
This form is useful: it shows the **vertex** 顶点 (turning point) of the curve at $(-p, q)$, and it gives a quick way to solve the equation.

Worked example. Write $9x^2 - 36x + 8$ in the form $p(x + q)^2 + r$.

Take the factor 9 out of the first two terms, then complete the square inside:

$$\begin{aligned}9x^2 - 36x + 8 &= 9(x^2 - 4x) + 8 \\&= 9((x - 2)^2 - 4) + 8 \\&= 9(x - 2)^2 - 36 + 8 = 9(x - 2)^2 - 28.\end{aligned}$$

So $p = 9$, $q = -2$, $r = -28$.



The completed square hands you the vertex: least value -28 at x equals 2

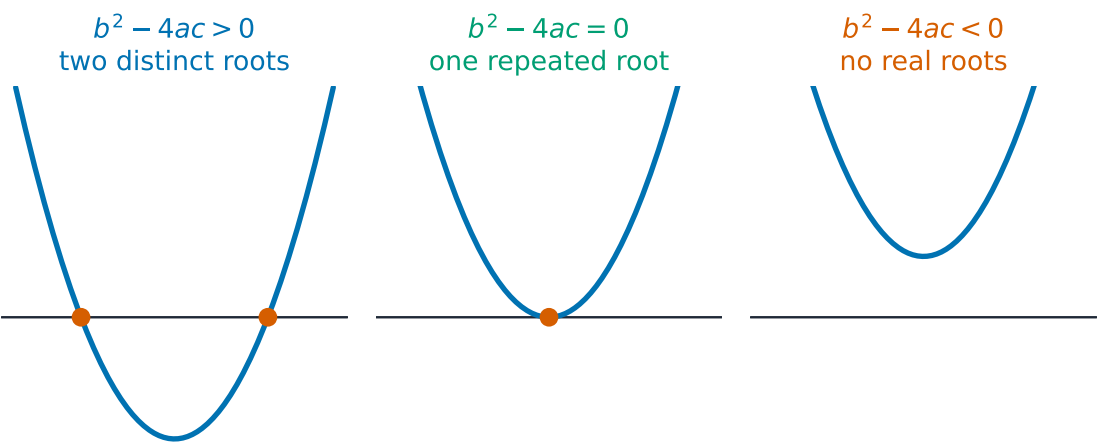
The discriminant

The **discriminant** 判别式 of $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac.$$

It tells you how many **real roots** 实根 (real solutions) the equation $ax^2 + bx + c = 0$ has:

Discriminant	Roots
$b^2 - 4ac > 0$	two distinct 相异 real roots
$b^2 - 4ac = 0$	one repeated real root
$b^2 - 4ac < 0$	no real roots



$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ (real roots need } b^2 - 4ac \geq 0)$$

The sign of $b^2 - 4ac$ decides how many times the parabola meets the x -axis.

Worked example. Find the values of the constant k for which $3kx^2 + (k + 8)x + 3 = 0$ has two distinct real roots.

Here $a = 3k$, $b = k + 8$, $c = 3$. For two distinct real roots you need $b^2 - 4ac > 0$:

$$(k + 8)^2 - 4(3k)(3) > 0 \Rightarrow k^2 + 16k + 64 - 36k > 0 \Rightarrow k^2 - 20k + 64 > 0.$$

Factorise: $(k - 4)(k - 16) > 0$, so $k < 4$ or $k > 16$. You also need $a \neq 0$, so $k \neq 0$.

Quadratic equations and inequalities

To solve a **quadratic equation** 二次方程, factorise, complete the square, or use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To solve a **quadratic inequality** 二次不等式 such as $(k - 4)(k - 16) > 0$, find the two roots, then decide which side of each root makes the statement true. A sketch of the parabola 抛物线 helps: the curve is above the x -axis (positive) outside the roots and below it (negative) between them.

Simultaneous equations

To solve a pair of **simultaneous equations** 联立方程 where one is linear and one is quadratic, use **substitution** 代入: rearrange the linear equation for one letter, then put that into the quadratic. This gives a single quadratic to solve.

Equations that are quadratic in disguise

Some equations are quadratic in **some function of x** . For example $x^4 - 5x^2 + 4 = 0$ is quadratic in x^2 : let $u = x^2$, solve $u^2 - 5u + 4 = 0$, then go back to x . You will use this idea again in trigonometry.

Functions

A **function** 函数 is a rule that sends each input to exactly one output. Write it as $f(x)$. The **composition of functions** 复合函数 $fg(x)$ means apply g first, then apply f to the result.

- The **domain** 定义域 is the set of allowed inputs x .
- The **range** 值域 is the set of outputs the function actually produces.

A function is **one-one** 一一对应 if different inputs always give different outputs. (No output is repeated.) Only a one-one function has an **inverse function** 反函数 f^{-1} , which reverses the rule.

The **composition** 复合 of two functions means doing one after the other. $fg(x)$ means "do g first, then f ": $fg(x) = f(g(x))$. The composite function fg exists only when the range of g lies inside the domain of f .

Finding an inverse

To find f^{-1} : write $y = f(x)$, make x the subject, then swap letters.

Worked example. The function $f(x) = (x + 3)^2 - 12$ is defined for $x \geq 0$. Find $f^{-1}(x)$.

Write $y = (x + 3)^2 - 12$ and solve for x :

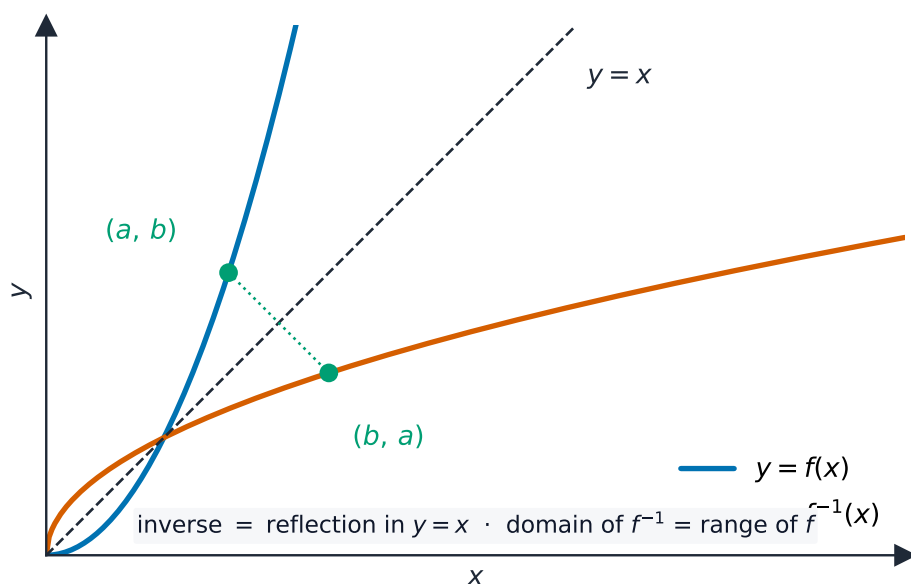
$$(x + 3)^2 = y + 12 \Rightarrow x + 3 = \sqrt{y + 12} \Rightarrow x = \sqrt{y + 12} - 3.$$

You take the **positive** square root because $x \geq 0$ means $x + 3 \geq 3 > 0$. So

$$f^{-1}(x) = \sqrt{x + 12} - 3.$$

Graphs of inverses and transformations

The graph of $y = f^{-1}(x)$ is the **reflection** 反射 of $y = f(x)$ in the line $y = x$.

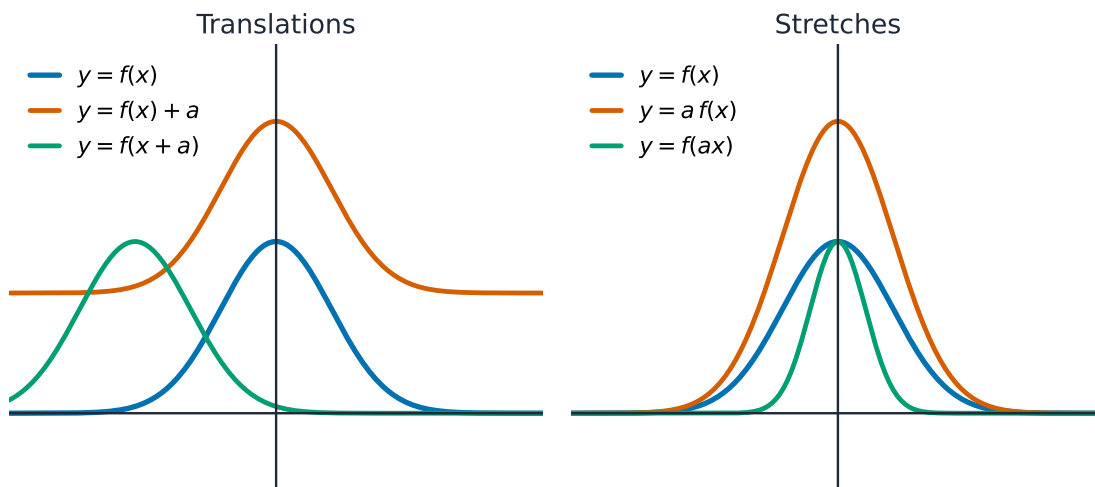


Reflecting $y = f(x)$ in the line $y = x$ gives its inverse; a point (a, b) becomes (b, a) .

You should know these **transformations** 变换 of $y = f(x)$:

New equation	Effect on the graph
$y = f(x) + a$	translation 平移 up by a
$y = f(x + a)$	translation left by a
$y = a f(x)$	stretch 伸缩 in the y -direction, scale factor a
$y = f(ax)$	stretch in the x -direction, scale factor $\frac{1}{a}$

When two transformations are combined, the order can matter. State each one fully (type, direction, and amount).



$f(x) + a$ moves up · $f(x + a)$ moves left · $af(x)$ stretch y · $f(ax)$ compress x if $a > 1$

Adding to the output or input slides the curve; a multiplier stretches it.

Coordinate geometry

Coordinate geometry 坐标几何 studies lines and circles using their equations.

Straight lines

The **gradient** 斜率 (steepness) of the line joining (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

You can write the **equation of a straight line** 直线方程 in any of these forms:

$$y = mx + c, \quad y - y_1 = m(x - x_1), \quad ax + by + c = 0.$$

Two lines are **parallel** 平行 when their gradients are equal, and **perpendicular** 垂直 (at right angles) when the product of their gradients is -1 .

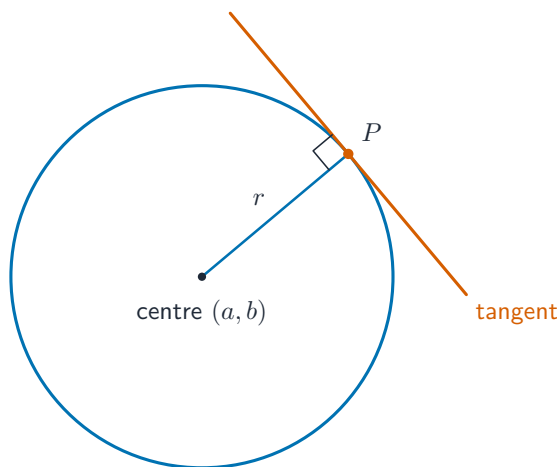
Circles

The **circle** 圆 with **centre** 圆心 (a, b) and **radius** 半径 r has equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

An expanded form like $x^2 + y^2 - 6x + 10y - 27 = 0$ is the same circle: complete the square in x and in y to find the centre and radius.

A **tangent** 切线 to a circle touches it at one point and is perpendicular to the radius at that point. This right-angle fact solves most circle problems.



A tangent touches the circle once and meets the radius at a right angle.

Worked example. The points $P(1, 1)$ and $Q(7, 11)$ are the ends of a **diameter** 直径 of a circle. Find the equation of the circle.

The centre is the midpoint of PQ :

$$\left(\frac{1+7}{2}, \frac{1+11}{2} \right) = (4, 6).$$

The radius is half the length of PQ :

$$r = \frac{1}{2} \sqrt{(7-1)^2 + (11-1)^2} = \frac{1}{2} \sqrt{36 + 100} = \frac{1}{2} \sqrt{136} = \sqrt{34}.$$

So the circle is $(x - 4)^2 + (y - 6)^2 = 34$.

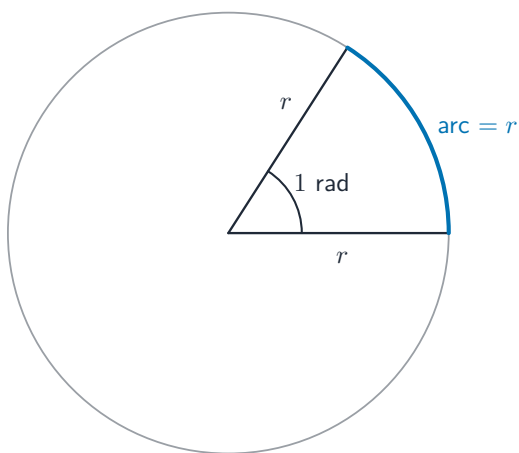
Circular measure

Radians

A **radian** 弧度 is another way to measure angles. One radian is the angle at the centre of a circle that cuts off an **arc** 弧 equal in length to the radius. The link between radians and **degrees** 度 is

$$\pi \text{ radians} = 180^\circ.$$

So to change degrees to radians, multiply by $\frac{\pi}{180}$; to change radians to degrees, multiply by $\frac{180}{\pi}$.



one **radian**: the angle whose arc is exactly one radius long

$$\pi \text{ rad} = 180^\circ, \text{ so } 1 \text{ rad} \approx 57.3^\circ$$

$$\text{degrees} \rightarrow \text{radians: } \times \frac{\pi}{180} \quad \text{radians} \rightarrow \text{degrees: } \times \frac{180}{\pi}$$

One radian: the angle whose arc equals the radius

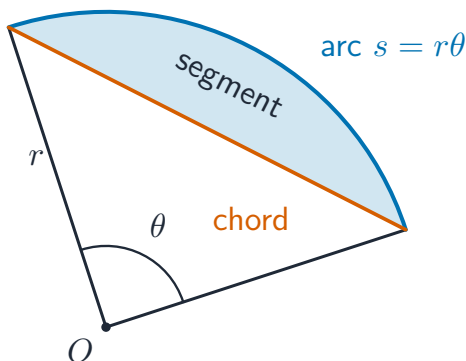
Arc length and sector area

For a **sector** 扇形 with radius r and angle θ in radians:

$$\text{arc length} = s = r\theta, \quad \text{sector area} = A = \frac{1}{2}r^2\theta.$$

A **chord** 弦 cuts the sector into a triangle and a **segment** 弓形. The segment area is the sector minus the triangle:

$$\text{segment} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta).$$



The shaded segment is the part of the sector between the chord and the arc.

Worked example. A sector has centre O and the angle at O is $\frac{2}{3}\pi$ radians. Show that the segment cut off by the chord has area about $0.614r^2$.

$$\text{segment} = \frac{1}{2}r^2 \left(\frac{2}{3}\pi - \sin \frac{2}{3}\pi \right) = \frac{1}{2}r^2 (2.0944 - 0.8660) = \frac{1}{2}r^2 (1.2284) \approx 0.614r^2.$$

Trigonometry

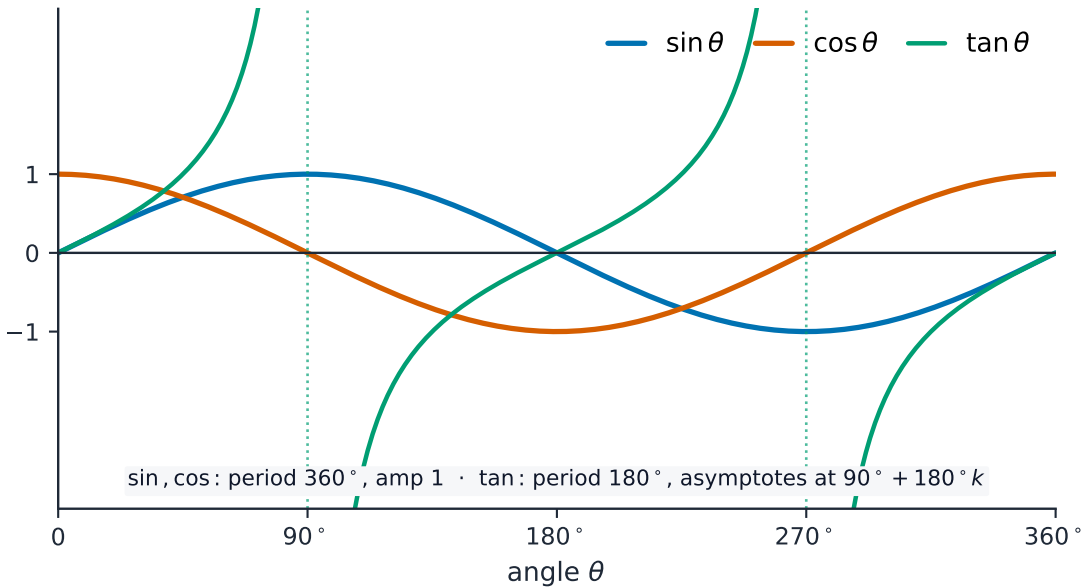


A Ferris wheel: a point on the rim rises and falls like a sine curve.

Graphs and exact values

You must know the shape of the graphs of the **sine** 正弦, **cosine** 余弦 and **tangent function** 正切 (written sin, cos, tan). The sine and cosine graphs wave between -1 and 1 and repeat every 360° (2π). Learn these exact values:

θ	30°	45°	60°
$\sin \theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos \theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan \theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$



Over one turn sin and cos stay between -1 and 1 ; tan shoots off at 90° and 270° .

The notations $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ mean the **inverse** angle (the **principal value** 主值).

Identities

An **identity** 恒等式 is true for every value of the angle. The two you must know are

$$\tan \theta \equiv \frac{\sin \theta}{\cos \theta}, \quad \sin^2 \theta + \cos^2 \theta \equiv 1.$$

Use them to rewrite an equation so it contains only one trig function.

Solving trigonometric equations

To solve a **trigonometric equation** 三角方程, first reduce it to one function, then find every solution in the given interval.

Worked example. Solve $6 \sin \theta = 1 + \frac{2}{\sin \theta}$ for $-180^\circ < \theta < 180^\circ$.

Multiply through by $\sin \theta$ to clear the fraction. This makes a **quadratic in $\sin \theta$** :

$$6 \sin^2 \theta - \sin \theta - 2 = 0 \Rightarrow (3 \sin \theta - 2)(2 \sin \theta + 1) = 0.$$

So $\sin \theta = \frac{2}{3}$ or $\sin \theta = -\frac{1}{2}$.

- $\sin \theta = \frac{2}{3}$: $\theta = 41.8^\circ$ or $\theta = 180^\circ - 41.8^\circ = 138.2^\circ$.
- $\sin \theta = -\frac{1}{2}$: $\theta = -30^\circ$ or $\theta = -150^\circ$.

The four solutions are $\theta = -150^\circ, -30^\circ, 41.8^\circ, 138.2^\circ$.

Series

The binomial expansion

For a positive integer n , the **binomial expansion** 二项展开式 is

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + b^n,$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is a **binomial coefficient** 二项式系数.

Worked example. Find the first three terms, in ascending powers of x , of $(2 - px)^5$.

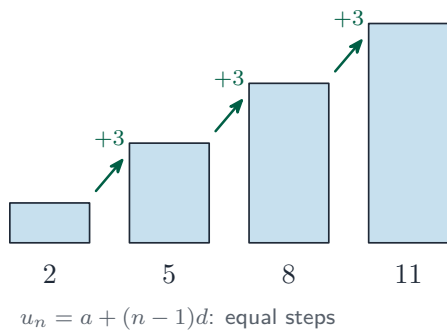
$$(2 - px)^5 = 2^5 + \binom{5}{1} 2^4 (-px) + \binom{5}{2} 2^3 (-px)^2 + \cdots = 32 - 80px + 80p^2x^2 + \cdots$$

Arithmetic and geometric progressions

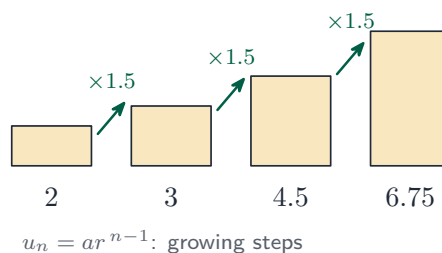
A **progression** 数列 (sequence) is a list of terms following a rule.

- An **arithmetic progression** 等差数列 (AP) adds a fixed **common difference** 公差 d each step. The n th **term** 项 is $u_n = a + (n-1)d$, and the sum of the first n terms is $S_n = \frac{n}{2}(2a + (n-1)d)$.
- A **geometric progression** 等比数列 (GP) multiplies by a fixed **common ratio** 公比 r each step. The n th term is $u_n = ar^{n-1}$, and $S_n = \frac{a(1-r^n)}{1-r}$.

arithmetic: add d each step



geometric: multiply by r each step



An AP climbs in equal steps; a GP's steps grow by the same ratio

A GP is **convergent** —it **converges** 收敛 (settles to a limit) —when $|r| < 1$. Then it has a **sum to infinity** 无穷和

$$S_{\infty} = \frac{a}{1 - r}.$$

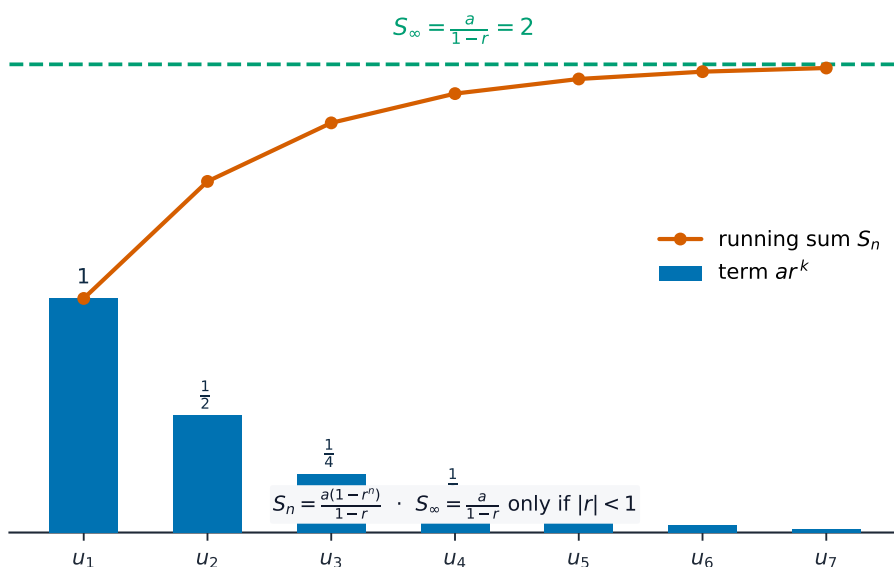
Worked example. The third term of a GP is 18 and the sum of the first three terms is 26. The common ratio is negative. Find the sum to infinity.

From $ar^2 = 18$ you get $a = \frac{18}{r^2}$. Put this into $a(1 + r + r^2) = 26$:

$$18(1 + r + r^2) = 26r^2 \Rightarrow 8r^2 - 18r - 18 = 0 \Rightarrow (4r + 3)(r - 3) = 0.$$

The ratio is negative, so $r = -\frac{3}{4}$ and $a = \frac{18}{(3/4)^2} = 32$. Then

$$S_{\infty} = \frac{32}{1 - (-\frac{3}{4})} = \frac{32}{\frac{7}{4}} = \frac{128}{7}.$$



*Convergent GP: bars shrink toward zero while the running sum approaches S infinity
 $= a/(1-r)$*

Differentiation

Differentiation 微分 finds the **gradient of a curve** 曲线斜率 at each point. The gradient is the **limit** 极限 of the gradients of shorter and shorter **chords**, called the **derivative** 导数.

The rules

Write the derivative as $f'(x)$ or $\frac{dy}{dx}$. The basic rule is

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{for any rational } n.$$

Differentiate sums term by term, and use the **chain rule** 链式法则 for a function inside a function:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

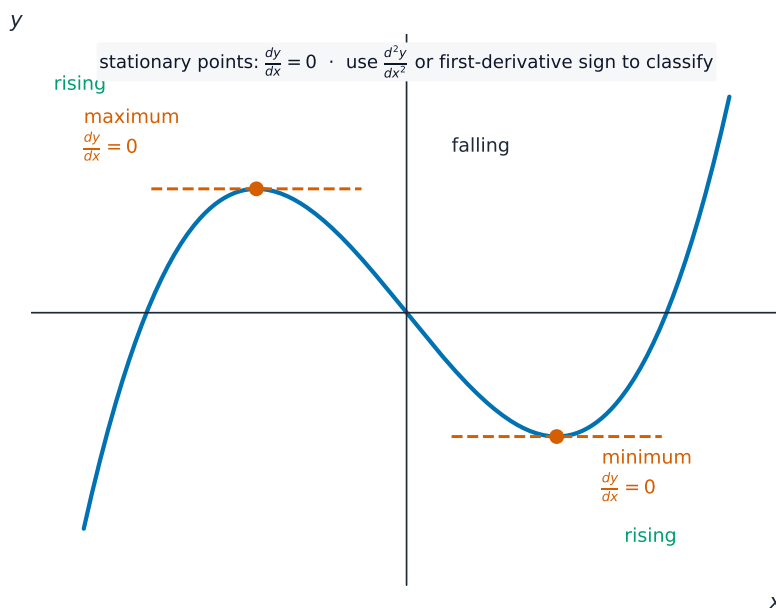
Differentiating again gives the **second derivative** 二阶导数 $f''(x)$ or $\frac{d^2y}{dx^2}$.

Using the derivative

- **Tangent and normal.** The gradient of the curve at a point is the gradient of the **tangent** there. The **normal** 法线 is perpendicular to the tangent, so its gradient is $-\frac{1}{(\text{tangent gradient})}$.
- **Increasing or decreasing.** The function is an **increasing function** 增函数 where $\frac{dy}{dx} > 0$, and a **decreasing function** 减函数 where $\frac{dy}{dx} < 0$.
- **Rate of change.** A derivative is a **rate of change** 变化率. Linked rates use the chain rule, e.g. $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$.

Stationary points

A **stationary point** 驻点 is where $\frac{dy}{dx} = 0$. Test its nature with the second derivative: $f''(x) > 0$ gives a **minimum point** 极小值点, and $f''(x) < 0$ gives a **maximum point** 极大值点. When $f''(x) = 0$ the test is inconclusive: the point may be a **point of inflexion** 拐点, where the curve changes concavity (the way it bends) – check the sign of $\frac{dy}{dx}$ just before and after to decide.



At a maximum or a minimum the tangent is flat, so $\frac{dy}{dx} = 0$.

Worked example. The curve $y = 4x^{1/2} - x$ has a maximum point at $x = a$. Find a .

$$\frac{dy}{dx} = 2x^{-1/2} - 1 = 0 \Rightarrow \frac{2}{\sqrt{x}} = 1 \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4.$$

So $a = 4$.

Integration

Integration 积分 is the reverse of differentiation. Reversing the power rule gives

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C \quad (n \neq -1).$$

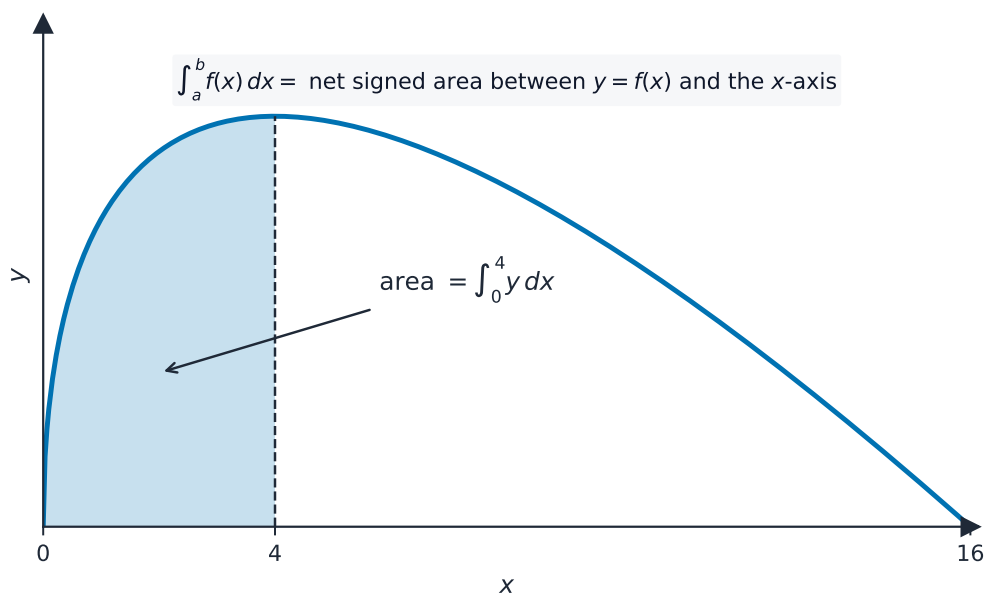
The $+C$ is the **constant of integration** 积分常数. If you know one point on the curve, substitute it to find C .

Definite integrals and area

A **definite integral** 定积分 has limits and gives a number:

$$\int_p^q f(x) dx = [F(x)]_p^q = F(q) - F(p).$$

The area of the **region** 区域 between a curve and the x -axis, from $x = p$ to $x = q$, is $\int_p^q y dx$. For the area between two curves, integrate (top curve $-$ bottom curve).



The definite integral $\int_0^4 y dx$ is the shaded area under the curve.

Worked example. The curve $y = 4x^{1/2} - x$ meets the x -axis again at $x = 16$. Find

the area between the curve and the x -axis from $x = 0$ to $x = 4$.

$$\int_0^4 (4x^{1/2} - x) dx = \left[\frac{8}{3}x^{3/2} - \frac{x^2}{2} \right]_0^4 = \frac{8}{3}(8) - \frac{16}{2} = \frac{64}{3} - 8 = \frac{40}{3}.$$

An **improper integral** 广义积分 has an infinite limit or an endpoint where the integrand is undefined; evaluate it as a **limit**. For example,

$$\int_1^\infty x^{-2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b} \right) = 1,$$

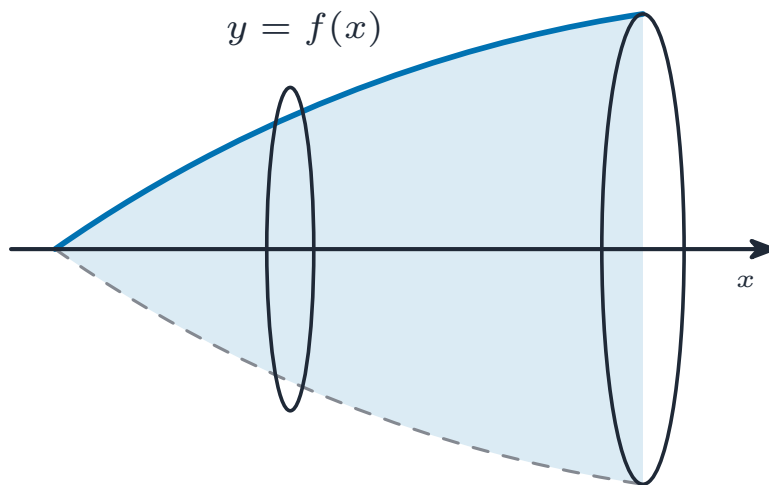
and $\int_0^1 x^{-1/2} dx = [2x^{1/2}]_0^1 = 2$, which is finite even though the integrand blows up at $x = 0$.

Volume of revolution

When a region is turned all the way around an axis it sweeps out a solid. The **volume of revolution** 旋转体体积 about the x -axis is

$$V = \pi \int_p^q y^2 dx,$$

and about the y -axis it is $V = \pi \int x^2 dy$. For example, the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$, turned about the x -axis, has volume $\pi \int_0^4 x dx = \pi \left[\frac{x^2}{2} \right]_0^4 = 8\pi$.



rotate the region around the x -axis to sweep a solid

Rotating the region under a curve around the x -axis sweeps out a solid; each thin slice is a disc of area πy^2 , so $V = \pi \int y^2 dx$

Exam tips

- Show **every line of algebra** —method marks are lost by jumping steps; use the **discriminant** $b^2 - 4ac$ to decide the number of real roots.
- Work in **radians** for circular measure (arc = $r\theta$, sector area = $\frac{1}{2}r^2\theta$) and for calculus of trig functions.
- For **differentiation**, set $\frac{dy}{dx} = 0$ for stationary points and use the second derivative to classify them.
- For **integration**, add $+c$ to an indefinite integral and use limits for area; an area below the axis gives a negative integral.