Further Mechanics

A-Level Further Mathematics

This handout covers Topic 3: **Further Mechanics** 进阶力学. It extends mechanics to projectiles, rigid bodies, circular motion, elastic strings, variable forces and collisions. Take $g = 10$ m s^{-2}.

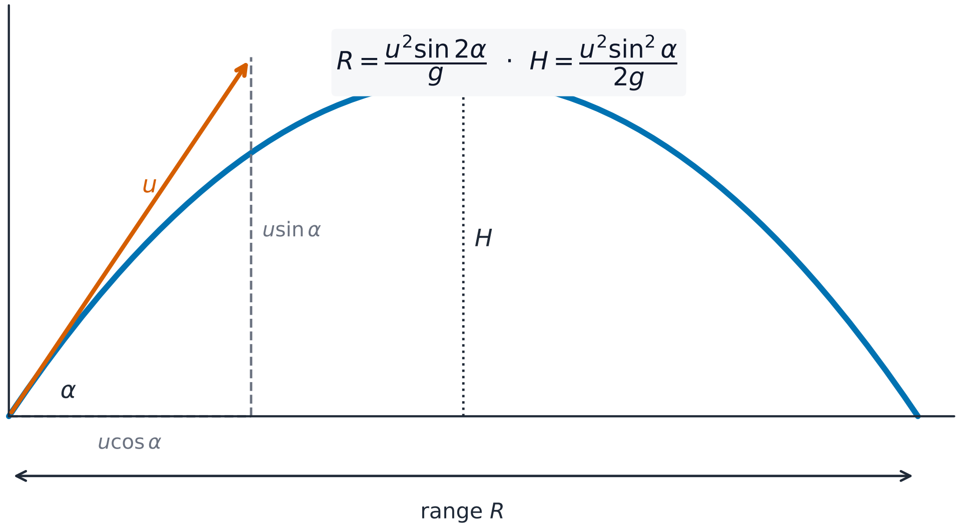
Motion of a projectile

Water jets follow parabolic paths —the classic projectile motion under gravity.

A **projectile** 抛射体 moves freely under gravity, so it has **constant acceleration** 匀加速 g downwards and no horizontal acceleration. Treat the horizontal and vertical motions separately. If it is launched at speed u and angle α:

$$\text{horizontal: } x = u\cos\alpha \cdot t, \qquad \text{vertical: } y = u\sin\alpha \cdot t - \tfrac{1}{2}gt^2.$$

Eliminating t gives the **Cartesian equation** 直角坐标方程 of the **trajectory** 轨迹 (the path), which is a parabola. For a fixed launch speed, all the trajectories lie under a **bounding parabola** 包络抛物线 (the envelope of reachable points). The range on level ground is $\dfrac{u^2 \sin 2\alpha}{g}$ and the greatest height is $\dfrac{u^2 \sin^2 \alpha}{2g}$.



The path is a parabola; the launch speed u splits into a steady horizontal part and a vertical part slowed by gravity.

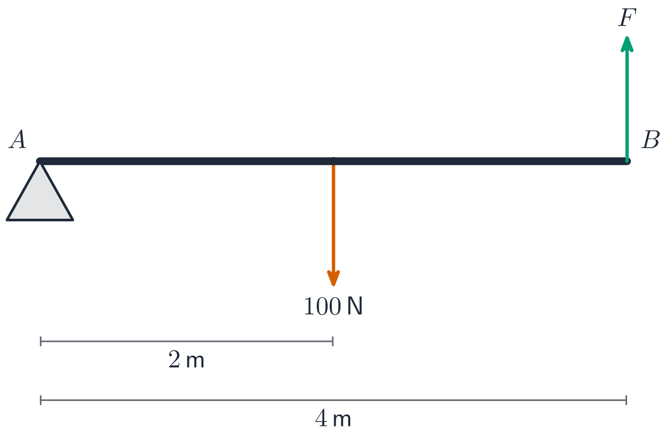
Worked example. A ball is thrown at $u = 20$ m s^{-1} at 30° to the horizontal. Find the range and greatest height.

$$\text{range} = \frac{20^2 \sin 60°}{10} = 40 \times 0.866 = 34.6 \text{ m}, \qquad \text{height} = \frac{20^2 \sin^2 30°}{2 \times 10} = \frac{400 \times 0.25}{20} = 5 \text{ m}.$$

Equilibrium of a rigid body

The **moment** 力矩 of a force about a point is force × perpendicular distance; it measures turning effect. The weight of a body acts at its **centre of mass** 质心, which you can find using **symmetry** 对称 for a uniform flat shape (a **lamina** 薄片), or by treating a composite body as a set of particles.

A rigid body under **coplanar forces** 共面力 is in **equilibrium** 平衡 when two conditions both hold: the vector sum of the forces is zero, and the sum of the moments about any point is zero. A body may also be on the edge of **toppling** 翻倒 (turning over) or **sliding** 滑动 (slipping).



Taking moments about the pivot A balances the turning effects: $F \times 4 = 100 \times 2$.

Worked example. A uniform beam AB of length 4 m and weight 100 N rests on a pivot at A. A vertical force F at B keeps it horizontal. Find F.

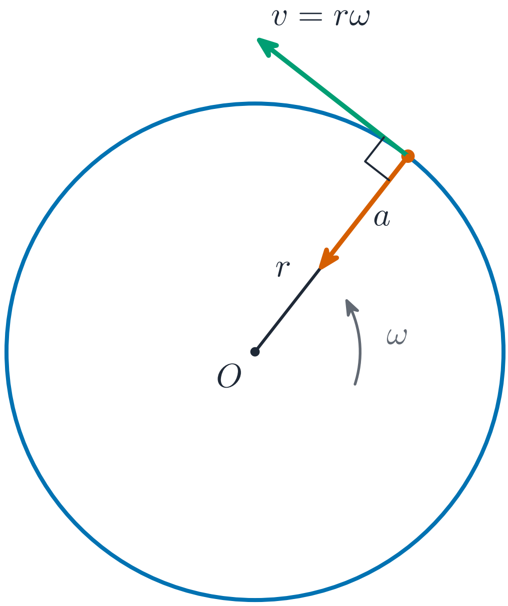
Take moments about A (the weight acts at the centre, 2 m from A):

$$F \times 4 = 100 \times 2 \ \Rightarrow \ F = 50 \text{ N}.$$

Circular motion

For a particle moving in a circle of radius r, the **angular speed** 角速度 ω links to the speed by $v = r\omega$. The acceleration points towards the centre —the **centripetal acceleration** 向心加速度—with size

$$a = r\omega^2 = \frac{v^2}{r}.$$



The velocity points along the tangent; the acceleration points inward to the centre.

The hammer thrower is a moving diagram of this page. The taut wire pulls the ball toward the centre —that inward pull is the centripetal force —while the ball's velocity points along the tangent. Let go, and with no inward force left the ball flies straight off that tangent

In a **horizontal circle** 水平圆 the speed is constant —as in a **conical pendulum** 圆锥摆 (a mass swung on a string). In a **vertical circle** 竖直圆 use energy conservation, because the speed changes with height; the **normal contact force** 法向接触力 or string tension provides the centripetal force.

Worked example. A particle moves in a horizontal circle of radius 2 m with angular speed 3 rad s^{-1}. Find its speed and acceleration.

$$v = r\omega = 2 \times 3 = 6 \text{ m s}^{-1}, \qquad a = r\omega^2 = 2 \times 3^2 = 18 \text{ m s}^{-2}.$$

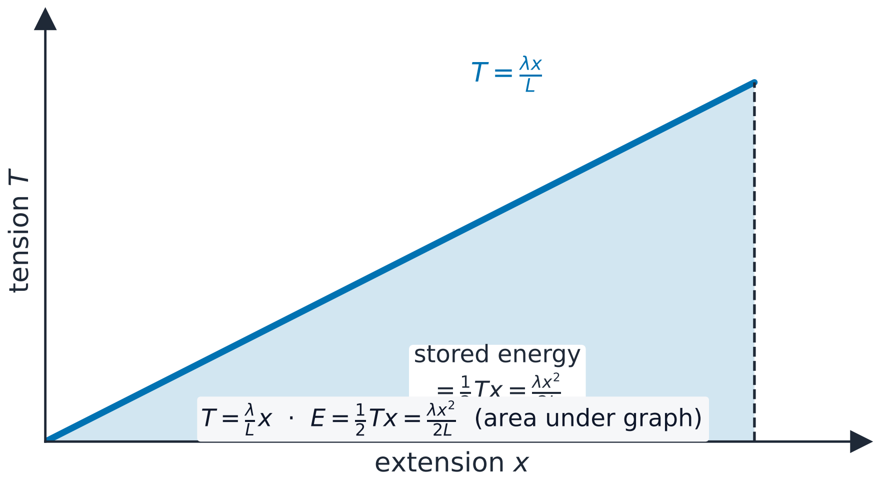
Hooke's law

Hooke's law 胡克定律 says the tension in an elastic string or spring is proportional to its extension x:

$$T = \frac{\lambda x}{L},$$

where L is the natural length and λ is the **modulus of elasticity** 弹性模量. Stretching stores **elastic potential energy** 弹性势能:

$$E = \frac{\lambda x^2}{2L}.$$



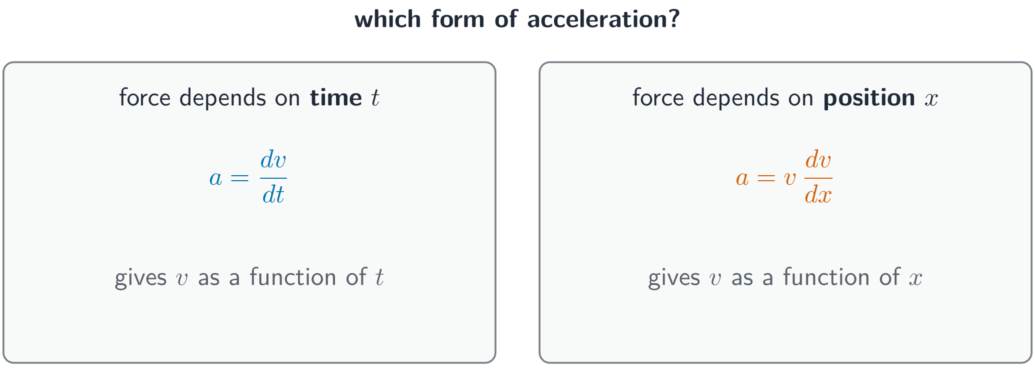
Tension rises in proportion to extension; the shaded triangle is the stored elastic energy.

Worked example. An elastic string of natural length 2 m and modulus 50 N is stretched by 0.5 m. Find the tension and the stored energy.

$$T = \frac{50 \times 0.5}{2} = 12.5 \text{ N}, \qquad E = \frac{50 \times 0.5^2}{2 \times 2} = 3.125 \text{ J}.$$

Linear motion under a variable force

When the force depends on position x, use acceleration in the form $a = v\dfrac{dv}{dx}$, which turns Newton's law into a **differential equation** 微分方程 relating v and x.



both come from $a = \dfrac{dv}{dt}$ with the chain rule; pick the one matching what the force depends on

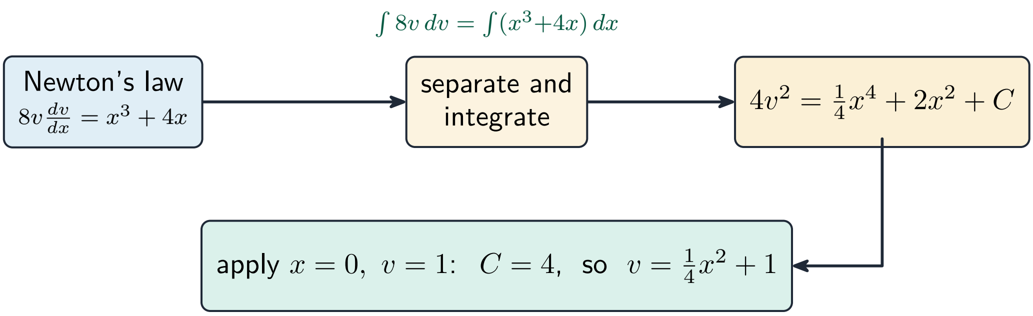
Use dv/dt when the force depends on time, v dv/dx when it depends on position

Worked example. A particle of mass 8 kg moves along a line under a **variable force** 变力 of size $(x^3 + 4x)$ N acting in the direction of motion. When $x = 0$, $v = 1$. Find v in terms of x.

Newton's law gives $8v\dfrac{dv}{dx} = x^3 + 4x$. Separating and integrating:

$$\int 8v\,dv = \int (x^3 + 4x)\,dx \;\Rightarrow\; 4v^2 = \tfrac{1}{4}x^4 + 2x^2 + C.$$

At $x = 0$, $v = 1$ gives $C = 4$, so $4v^2 = \frac{1}{4}x^4 + 2x^2 + 4 = 4\left(\frac{x^2}{4} + 1\right)^2$. Hence $v = \frac{1}{4}x^2 + 1$.



a position-dependent force becomes a **differential equation** in v and x

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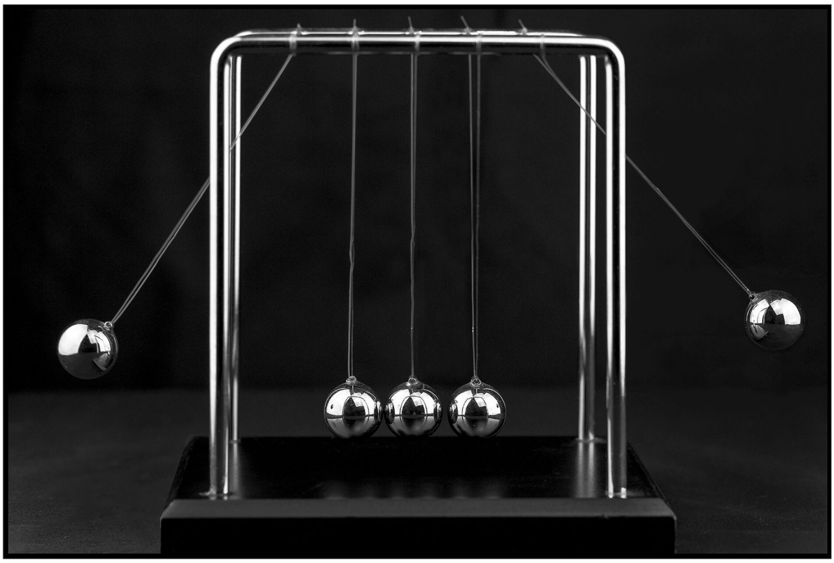
Momentum

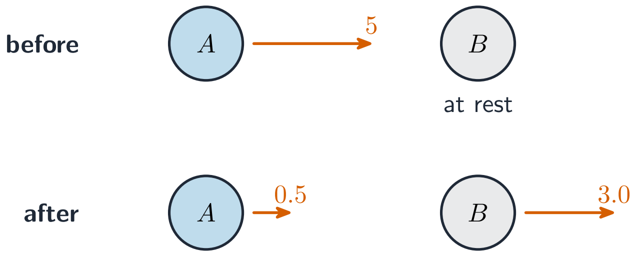


A Newton's cradle demonstrates conservation of momentum in collisions.

In a collision, total **linear momentum** is conserved —the **conservation of linear momentum** 动量守恒. The bounciness is measured by **Newton's experimental law** 牛顿实验定律, which defines the **coefficient of restitution** 恢复系数 e:

$$e = \frac{\text{speed of separation}}{\text{speed of approach}}, \qquad 0 \leqslant e \leqslant 1.$$

Here $e = 1$ is perfectly elastic (no energy lost) and $e = 0$ is inelastic (the bodies stay together). In an **oblique impact** 斜碰撞, resolve the velocities along and perpendicular to the line of impact, applying restitution along it.



$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{3.0 - 0.5}{5} = 0.5$$

The coefficient of restitution e compares how fast the bodies separate with how fast they approached.

Worked example. A sphere A of mass 2 kg moving at 5 m s^{-1} hits a stationary sphere B of mass 3 kg, with $e = 0.5$. Find the speeds afterwards.

Momentum: $2(5) = 2v_A + 3v_B$, so $2v_A + 3v_B = 10$. Restitution: $v_B - v_A = 0.5(5) = 2.5$. Solving together gives $v_A = 0.5$ m s^{-1} and $v_B = 3.0$ m s^{-1}.

Exam tips

- For **projectiles**, resolve into horizontal (constant velocity) and vertical ($a = g$) motion, linked by the same time.
- For a **rigid body in equilibrium**, take moments about a point that removes an unknown force.
- For **circular motion**, use $F = mv^2/r = m\omega^2 r$ towards the centre; in a vertical circle check the minimum speed at the top.
- With a **variable force**, use $a = v\,\frac{dv}{dx}$ and integrate; elastic PE $= \frac{\lambda x^2}{2L}$.