FURTHER MATHEMATICS

Paper 9231/11
Further Pure Mathematics 1

Key messages

Candidates should read each question carefully and check that they answer each part in sufficient detail. In particular, when proving a given result, all necessary steps must be shown.

When sketching a graph, all significant points should be labelled and the behaviour at extreme values should be checked.

Algebra can often be simplified by efficient use of brackets.

General comments

Candidates showed a good understanding of the material.

Candidates' algebra was accurate and most completed the paper in the time allowed.

Comments on specific questions

Question 1

(a) The correct formulae from MF19 were used and the expansion to the required form was usually accurate. Only a few did not simplify completely.

(b) Most candidates had a fully correct solution. Very few tried partial fractions of the wrong form to establish the given result.

(c) This almost always followed correctly from **part (b).**

Question 2

(a) & (b) Most candidates recognised the shear. Good responses gave a full description using a point; many stated a factor of 3/2 instead and did not receive full credit.

(c) Most candidates correctly calculated **M** and its determinant to show that area was unchanged by the transformation. A small number gave an argument based on the fact that a shear does not change area.

(d) This was almost always correct.

Question 3

Most candidates stated the base case and began the induction by successfully differentiating using the product rule and then showing that the formula matched their result.

The hypothesis was stated algebraically, and only a handful wrongly assumed it to be true for all values.

The result needed two stages of differentiation, and the best solutions simplified the result after the first step to make the second step less complicated. It was then more straightforward to show that their result matched the form of the result.

A minority of responses wrote a complete final statement, quoting the algebraic form given in the question and also saying that it has been shown to be true for all positive integers, as required.

Question 4

(a) This was well done, with almost all candidates using substitution and showing sufficient detail of the expansions to justify the given result.

(b) This was usually correct.

(c) Most candidates summed correctly over the roots and efficiently found the final answer. A small number used a second substitution, which needed more time and algebraic steps.

Question 5

Candidates are advised to check that they have copied vectors correctly. Errors in arithmetic, particularly in evaluating a cross product, can cause problems throughout a question.

(a) The majority correctly identified which were position vectors and which were direction vectors and efficiently found the equation of the plane. There were a few numerical errors in taking the cross product.

(b) Those who used the method of substituting the coordinates of the point into the plane usually remembered to first arrange the plane in the form $ax + by + cz - d = 0$.

Another popular and successful method was to use a parameter to find the foot of the line perpendicular to the plane.

(c) There were many good solutions; only a few forgot that the equation must be given as $\boldsymbol{r}$ =.

The usual method was to find the direction of the line using a cross product and find a common point by inspection. Another was to find two common points by inspection and use them to determine the direction vector.

Question 6

(a) There were many good diagrams, which made the pole and the initial line clear and gave the correct shape at the extremities.

(b) This was well done with most recognising that a use of a double angle formula was required. A few forgot to divide by 6 when integrating $\cos 6\theta$.

(c) Most candidates knew that they should consider $y = r\sin\theta$. Those who worked accurately to find the possible values of $\sin\theta$ sometimes gave the final answer as the value of r rather than the value of y.

(d) This was usually well done.

Question 7

(a) Almost all found the correct horizontal and vertical asymptotes.

(b) The differentiation was correctly done and the stationary values found.

(c) In most cases the asymptotes were drawn and labelled correctly and the approach to them was acceptable.

(d) The reflection in the x axis was usually correct, although candidates should check carefully that the shape of the reflected part matches their original graph. In particular, the point (–2, 0) should have shown a sharp change of direction and a change in curvature.

(e) The best solutions found the critical points using equations and then considered the graph to check that the final inequalities were correct. Many candidates did correctly cut out the points corresponding to the asymptotes from their final answer.

Candidates who attempted to use inequalities often multiplied their inequality by $x^2 + 3x + 1$ without considering that this could be negative and therefore change the inequality sign.

FURTHER MATHEMATICS

Paper 9231/12
Further Pure Mathematics 1

Key messages

Candidates should read every question very carefully so that they can answer all aspects in adequate depth.

Candidates should show all steps that lead to a solution, particularly when proving a given result.

Candidates need to be familiar with the formula sheet (MF19) so that they can use the results it contains.

Candidates should ensure that all sketched graphs are labelled fully and draw carefully to show significant points and behaviour at limits.

General comments

Most candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed a good understanding of linear algebra. There was some inaccuracies in work on inequalities particularly for **Questions 3(a)**, **3(b)** and **7(e)**. There were also many creative solutions to **4(b)**.

Comments on specific questions

Question 1

(a) There were many candidates who correctly applied the formula from MF19 to this question to gain a correct expression, however many candidates did not factorise at the end of the answer as required.

(b) Nearly all candidates were able to correctly express the fraction in the correct partial fractions. Many candidates did not show the sufficient number of complete terms to show cancelling out, with candidates needing to show at least one term cancelling out completely to know what was happening with the terms. Candidates generally presented their work neatly on the page.

(c) Most candidates with an answer to **1(b)** in a correct form were able to follow through correctly for this part.

Question 2

(a) Nearly all candidates completed this part correctly.

(b) This part was often well done, but when candidates tried to use a formula rather than the summation method they need to be sure the formula is correct.

Question 3

(a) As the base case was given in the question, some candidates struggled with this and wanted to start at $n = 2$ and moved everything up by 1. Some candidates struggled to distinguish between what they were to prove $u_n < 5$ and the definition of the sequence. There was some mistakes in inequalities work within some candidates' responses.

(b) There was some excellent, well explained work with the difference, considering every bracket in the fraction.

Question 4

(a) (i) This part was generally completed well by all candidates.

(ii) This part was generally completed well by all candidates with the most common error being them missing the centre of enlargement.

(b) There was a considerable amount of work expended by most candidates working with the co-factors to determine that the matrix was singular. Some candidates used the more efficient method of row operations to determine the matrix was singular. There were some very well laid out solutions.

Question 5

(a) This part was usually well answered with most responses being fully correct. The most common error was not explicitly stating the maximum distance.

(b) This part was successfully completed by candidates if they recognised the function to integrate. Common mistakes included omitting the half from the formula for the area and not integrating correctly, missing off the negative in their integrated function.

(c) This part was well done by most with a significant number of candidates writing $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$ and then into various forms.

(d) Many candidates who had scored well on **5(b)** did not seem able to find the area of the triangle and see the link between the parts or recognise the need to use the double angle formulae.

Question 6

(a) Most candidates were successful in this part. The errors were in the evaluation of the square root or in the use of the formula. Most candidates left it as an exact answer.

(b) This part was done well. The errors were slips in the evaluation of the cross product, and in forming the dot product to work out the angle. In addition, some candidates proceeded with extra working by subtracting their angle from 90 degrees.

(c) Most candidates went for an approach using two parameters, with a few introducing a third parameter as a multiple of the common normal direction. Mistakes within arithmetic and the omission of **r**= were common.

Question 7

(a) This part was successfully completed by most candidates. The most common error was in obtaining the oblique asymptote from an error in their algebraic manipulation.

(b) This part was successfully completed by most candidates. Common errors were in their differentiation, and solving the derivative set equal to zero. A few candidates did not write down the coordinates as requested but just the values of *x*.

(c) There were many good sketches by candidates of the graphs and the approaches to the asymptotes. Some candidates did not label all asymptotes clearly. A few candidates did not use the fact that they had just found the stationary value (0, 1) and instead put the lowest point off the axis.

(d) There were some very good sketches to this part. Common errors seen were that the graph was reflected in the *y*-axis and that the graph was not smooth at (0, 1).

(e) There were many correct solutions to this part. A considerable number of candidates continue to work with inequalities in this type of question which can lead to errors. Many candidates were prepared to work with $1-\sqrt{3}$ in effect for $|x|$. There were many two-part inequalities as final answers.

FURTHER MATHEMATICS

Paper 9231/13
Further Pure Mathematics 1

Key messages

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When sketching a graph, all significant points should be labelled and the behaviour at extreme values should be checked.

Algebra can often be simplified by efficient use of brackets.

General comments

Candidates showed a good understanding of the material.

Candidates' algebra was accurate and most completed the paper in the time allowed.

Comments on specific questions

Question 1

(a) The correct formulae from MF19 were used and the expansion to the required form was usually accurate. Only a few did not simplify completely.

(b) Most candidates had a fully correct solution. Very few tried partial fractions of the wrong form to establish the given result.

(c) This almost always followed correctly from **part (b).**

Question 2

(a) & (b) Most candidates recognised the shear. Good responses gave a full description using a point; many stated a factor of 3/2 instead and did not receive full credit.

(c) Most candidates correctly calculated **M** and its determinant to show that area was unchanged by the transformation. A small number gave an argument based on the fact that a shear does not change area.

(d) This was almost always correct.

Question 3

Most candidates stated the base case and began the induction by successfully differentiating using the product rule and then showing that the formula matched their result.

The hypothesis was stated algebraically, and only a handful wrongly assumed it to be true for all values.

The result needed two stages of differentiation, and the best solutions simplified the result after the first step to make the second step less complicated. It was then more straightforward to show that their result matched the form of the result.

A minority of responses wrote a complete final statement, quoting the algebraic form given in the question and also saying that it has been shown to be true for all positive integers, as required.

Question 4

(a) This was well done, with almost all candidates using substitution and showing sufficient detail of the expansions to justify the given result.

(b) This was usually correct.

(c) Most candidates summed correctly over the roots and efficiently found the final answer. A small number used a second substitution, which needed more time and algebraic steps.

Question 5

Candidates are advised to check that they have copied vectors correctly. Errors in arithmetic, particularly in evaluating a cross product, can cause problems throughout a question.

(a) The majority correctly identified which were position vectors and which were direction vectors and efficiently found the equation of the plane. There were a few numerical errors in taking the cross product.

(b) Those who used the method of substituting the coordinates of the point into the plane usually remembered to first arrange the plane in the form $ax + by + cz - d = 0$.

Another popular and successful method was to use a parameter to find the foot of the line perpendicular to the plane.

(c) There were many good solutions; only a few forgot that the equation must be given as $\boldsymbol{r} =$.

The usual method was to find the direction of the line using a cross product and find a common point by inspection. Another was to find two common points by inspection and use them to determine the direction vector.

Question 6

(a) There were many good diagrams, which made the pole and the initial line clear and gave the correct shape at the extremities.

(b) This was well done with most recognising that a use of a double angle formula was required. A few forgot to divide by 6 when integrating $\cos 6\theta$.

(c) Most candidates knew that they should consider $y = r\sin\theta$. Those who worked accurately to find the possible values of $\sin\theta$ sometimes gave the final answer as the value of r rather than the value of y.

(d) This was usually well done.

Question 7

(a) Almost all found the correct horizontal and vertical asymptotes.

(b) The differentiation was correctly done and the stationary values found.

(c) In most cases the asymptotes were drawn and labelled correctly and the approach to them was acceptable.

(d) The reflection in the x axis was usually correct, although candidates should check carefully that the shape of the reflected part matches their original graph. In particular, the point (–2, 0) should have shown a sharp change of direction and a change in curvature.

(e) The best solutions found the critical points using equations and then considered the graph to check that the final inequalities were correct. Many candidates did correctly cut out the points corresponding to the asymptotes from their final answer.

Candidates who attempted to use inequalities often multiplied their inequality by $x^2 + 3x + 1$ without considering that this could be negative and therefore change the inequality sign.

FURTHER MATHEMATICS

Paper 9231/14
Further Pure Mathematics 1

Key messages

Candidates should read each question carefully and check that they have answered each part in sufficient detail. In particular, when proving a given result, all necessary steps must be shown.

When sketching a graph all significant points should be labelled and the behaviour at extreme values checked.

Algebra can often be simplified by efficient use of brackets.

Inequalities need to be handled carefully. When multiplying an inequality candidates should always check that the multiplier is positive.

General comments

Candidates showed a good understanding of the material and most completed the paper in the time allowed.

They showed confident use of algebra and much of the work was very well presented.

Comments on specific questions

Question 1

The base case and hypothesis were generally well presented.

For the inductive step the most successful responses recognised that $\left(\frac{6}{5}\right)^{k+1} \geq \frac{6}{5}\left(1 + \frac{1}{5}k\right)$ and went on to show that this implies the required inequality for the $(k + 1)$th term. Those who considered the difference between the $(k + 1)$th and the kth term often had difficulty with the correct use of inequalities. Some did not use the hypothesis in their working.

The conclusion was well formed in most cases with the nth term stated correctly.

Question 2

(a) & (b) Both parts were generally well done. Many candidates worked back from the given expressions to obtain the result. Those that worked with $\sum\alpha$, $\sum\alpha\beta$ and $\alpha\beta\gamma$ and compared them with b, c and d communicated their approach clearly.

There were a few errors with signs of the coefficients.

(c) (i) This was well done, with almost all remembering to discard the negative solution of the quadratic equation.

(ii) This part was almost always done correctly.

Question 3

(a) This was done well with only a few expanding the brackets incorrectly or forgetting to sum the constant term.

(b) The partial fractions were usually found correctly – those with incorrect fractions would find that the terms did not cancel properly for the next stage.

Fully correct solutions showed at least one complete cancellation, which required all terms for the first three values of r to be shown. At least the last two values of r were also needed to make sure that the final answer did not omit any terms. Many candidates did not show enough terms and this often led to an incorrect final answer.

(c) This was almost always followed correctly from **(b)**.

Question 4

Candidates are advised to check that they have copied vectors correctly. Errors in arithmetic can also cause problems throughout a question, particularly when evaluating a cross product.

(a) Most candidates found correct direction vectors and used the cross-product method to find the equation of the plane.

(b) Almost all began by finding the direction of the common perpendicular, with a few errors in arithmetic. Most then used a correct vector to calculate the length required.

Question 5

(a) (i) & **(ii)** The addition formulae were used correctly.

The answer was given and so candidates needed to show the surds corresponding to each trigonometric ratio before multiplying them and rearranging to the final answer. This step was often omitted.

(b) The transformations were correctly identified. The most common error was to omit giving the centre of the rotation as the origin.

(c) The inverses of both matrices needed to be written down in the correct order to give the inverse transformation. Either surd or trigonometric form was acceptable for the answer.

The most common errors were in finding the determinant, particularly when the elements were given in surd form rather than as trigonometric functions.

A few candidates calculated **M** and then its inverse.

(d) The majority calculated **M** correctly and made it clear they were looking for invariant lines rather than invariant points. The quadratic equation in m was given and sufficient working needed to be shown to justify it fully.

The last part of the question was less familiar. The best solutions simply used the formula to solve the quadratic equation and then trigonometric identities to reduce the answer to the required form.

A particularly neat method was to write the equation as $m^2 + 2m\cot\frac{\pi}{12} - 1 = 0$ before solving.

Question 6

(a) There were many good graphs. Usually the single arc was correct, with some having difficulty with the shape at the extremities.

(b) Most candidates used a double angle formula, the most frequent error being to omit $\frac{1}{2}$ or to use a wrong sign in the formula.

(c) Nearly all knew that they needed to consider $y = r\sin\theta$ and differentiated it correctly.

There were several routes towards forming an equation in one trigonometric ratio, and there were many accuracy errors such as the omission of $\frac{1}{2}$, inclusion of an extra 2, or with signs.

When they had found a value for, for example, $\cos\left(\frac{1}{2}\theta\right)$, many candidates gave the corresponding value of r as their final answer rather than the value of y that was needed.

Question 7

(a) This was usually correct with most candidates using long division to find the oblique asymptote.

(b) This was well done with a few candidates making accuracy errors. Almost all candidates aimed to show that the derivative cannot equal 0, either by considering and evaluating the discriminant of an equation, or by rearrangement of the derivative.

(c) The graph was well drawn although not all candidates labelled the asymptotes and a few graphs did not approach the asymptotes correctly.

The intersections with the axis were found by most candidates.

(d) The majority of candidates were able to reflect the graph. Some needed to check that their new graph still approached the vertical asymptote correctly and had a sharp change of direction at the intersections with the x axis.

(e) The most efficient solutions recognised that they were looking for the values of x where $\frac{10x^2 - 11x - 18}{10x - 18} = 4$, solved the quadratic equation and identified which value corresponded to p and which to q. Some went on to consider the equation with –4 to check that this gave the complete inequality given in the question.

Problems arose for those who multiplied the inequality by $10x - 18$ without considering the change of inequality sign when $10x - 18 < 0$.

FURTHER MATHEMATICS

Paper 9231/21
Further Pure Mathematics 2

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.
- Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. Sometimes candidates did not fully justify their answers and jumped to conclusions without justification, particularly where answers were given within the question. There were many scripts of a high standard.

Comments on specific questions

Question 1

This was well done with the majority of candidates setting the determinant to zero and arriving at the correct values of *k*. The few errors seen were mainly due to accuracy or setting the determinant not equal to zero.

Question 2

(a) Almost all applied the chain rule to find the first derivative.

(b) Good candidates used the product rule to differentiate the first derivative with respect to t and applied the chain rule again to obtain $\dfrac{1-\sinh t}{\cosh^3 t}$.

(c) Most candidates maintained accuracy when subbing in to find the Maclaurin's series and simplified their answer fully.

Question 3

(a) The majority of candidates accurately differentiated the product given in the question. Those who then replaced x^2 with x^2+1-1 successfully derived the given reduction formula. Others who integrated $x^2\left(1+x^2\right)^{n-1}$ by parts were unsuccessful.

(b) Most candidates correctly applied the reduction formula from **3(a)** and used I_{-1} as their initial value. Those who tried to use I_{-3} as their initial value were unsuccessful.

Question 4

Almost all knew how to approach this question and completed it to a high standard. There were some inaccuracies when solving linear equations to find the particular integral and some errors when substituting initial conditions. A few candidates gave expressions instead of equations as their answer.

Question 5

(a) Most candidates formed a correct expression for the sum of the areas of the rectangles and good responses applied the sum of a geometric series, making clear the first term and common ratio, to accurately derive the given result. Errors included incorrect areas and numbers of terms.

(b) Most candidates correctly adapted their solution to **5(a)** and derived a suitable upper bound. There were some difficulties when simplifying the algebraic expressions which involved different exponents. Final responses were often in a variety of forms.

(c) Good candidates showed clearly that the difference between U_n and L_n is proportional to $\frac{1}{n}$, leading efficiently to the given answer.

Question 6

(a) Almost all read the diagonal of $\mathbf{P}$ and stated the eigenvalues.

(b) (i) Almost all maintained accuracy when squaring $\mathbf{P}$.

(ii) Most manipulated the characteristic equation by multiplying through by $\mathbf{P}^{-1}$. Good candidates accurately substituted $\mathbf{P}^2$ when making $\mathbf{P}^{-1}$ the subject.

(c) Most candidates applied $\mathbf{A} = \mathbf{PDP}^{-1}$ using their answer to **6(b)(ii)**. A few candidates took the lengthy approach of finding $\mathbf{P}^{-1}$ again without using the characteristic equation.

Question 7

(a) This was done successfully with candidates either differentiating the logarithmic form of $\tanh^{-1} x$ or differentiating implicitly.

(b) Almost all divided through by x and found the integrating factor. Good responses maintained accuracy and fully simplified the right hand side of the equation after multiplying by the integrating factor. Using integration by parts, most found $\frac{1}{3}x^3 + \frac{1}{3}\left(x^2+16\right)^{\frac{3}{2}} + C$ as the integral of the right hand side and substituted the initial conditions to find the correct value of C.

Question 8

(a) Most candidates stated the formula for the sum of the geometric progression accurately. A few wrote the exponent as n, which hindered their progress in **8(b)**. A few incorrectly stated that the sum was equal to zero.

(b) Almost all knew that de Moivre's theorem related the series to the sum of the geometric progression in **8(a)**. The strongest candidates took the imaginary part, after multiplying the numerator and denominator by the conjugate of z – 1. Use of $z = \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}$ or appropriate trigonometric identities then led to the given answer.

(c) Only the very strongest candidates differentiated the equation given in **8(a)** to find $\sum_{m=1}^{n}\left(\frac{1}{2}\right)^m m\cos m\theta$ in terms of n and θ.

(d) The few candidates who had a correct expression for **8(c)** were usually able to apply the given limit successfully.

FURTHER MATHEMATICS

Paper 9231/22
Further Pure Mathematics 2

Key messages

Candidates should read every question very carefully so that they can answer all aspects in adequate depth.

Candidates should show all steps that lead to a solution, particularly when proving a given result.

Candidates need to be familiar with the formula sheet (MF19) so that they can use the results it contains.

Candidates should ensure that all sketched graphs are labelled fully and draw carefully to show significant points and behaviour at limits.

General comments

Most candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed a good understanding of linear algebra.

Comments on specific questions

Question 1

This question was completed successfully by most candidates. Common errors in this questions included using the wrong formula for the arc length of a curve in polar form, or incorrectly squaring the expression by squaring the exponents instead of doubling them.

Question 2

This question was completed successfully by most candidates with them being able to rewrite the complex number from rectangular to exponential form. Common errors included failing to give the argument in the required range, and a small number of candidates forgetting to include the i in the exponential form.

Question 3

(a) This part was completed successfully by most candidates. Some candidates expanded $\left(x+y\right)^3$ before differentiating, this made the work unnecessarily complicated.

(b) There was some excellent, well set out work in response to this part. This part however was less well done than **3(a)** with candidates who expanded $\left(x+y\right)^2\left(1+y'\right)$ often losing terms during implicit differentiation.

Question 4

(a) This part was generally completed well by all candidates. The common error was candidates not grouping together $\left(z^n+z^{-n}\right)$ terms.

(b) Most candidates obtained the correct answer, but many forgot to substitute $\theta = 2x$ or used an incorrect limit after substitution which result in marks being lost.

Question 5

(a) This was a very well answered part, with most candidates providing fully correct responses. The common errors seen were that the candidates did not use the correct variables *x* and *y* or they did not have *y* = included for the final answer.

(b) Most candidates correctly used Pythagoras' theorem to find the modulus. Some candidates attempted to memorise a formula for the argument and wrote $\arctan 2$ instead of $\arctan\frac{1}{2}$. Those who applied the compound angle formula generally found out the argument correctly.

Question 6

(a) This part was successfully completed by most candidates. Some candidates applied the substitution and the hyperbolic identity successfully; however they omitted to replace the substitution variable *u* back to the original variable *x*.

(b) This part proved challenging for many candidates, however there were some excellently laid out solutions. Many candidates did not recognise the connection between the sum of rectangles and the left hand side of the series. Among those who formulated the sum correctly, few showed sufficient detail in manipulating the *n* to the answer given.

(c) This part proved challenging to many candidates with a significant number not identifying and using the correct rectangles to start their solution.

(d) This part was generally well answered by most candidates.

Question 7

(a) This part was generally well answered by most candidates. Most candidates correctly applied the product rule and the formula for the derivative for arcsine. However, some candidates could not complete the required simplification involving three terms with negative exponents.

(b) This part was successfully completed by many candidates. Most candidates found the integrating factor. However, many candidates did not recognise the link between **7(a)** and the integral. Some candidates also forgot to evaluate the constant $\arcsin\left(\frac{1}{2}\right)$ as $\frac{\pi}{6}$.

Question 8

(a) This part was successfully completed by nearly all candidates. The most common error was in evaluating the determinant correctly with sign errors being the main cause of incorrect values.

(b) This part proved challenging to candidates. Most candidates struggled to understand how to prove that a system was inconsistent. The geometrical interpretation also caused problems, however many candidates could have fortuitously arrived at the correct answer through a limited number of learnt interpretations.

(c) This part proved challenging to candidates. Most candidates were not able to prove that the system was inconsistent. In this part most candidates were able to recognise that the system represented two parallel planes, but many failed to state explicitly that the planes are not identical.

(d) This part was generally completed successfully. Most candidates successfully applied the Cayley-Hamilton substitution. One common error was that the matrix equation $\mathbf{A}^3 = 3\mathbf{A}^2 - 8\mathbf{A}$ cannot be cancelled by the matrix A.

FURTHER MATHEMATICS

Paper 9231/23
Further Pure Mathematics 2

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.
- Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. Sometimes candidates did not fully justify their answers and jumped to conclusions without justification, particularly where answers were given within the question. There were many scripts of a high standard.

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(c) Most candidates maintained accuracy when subbing in to find the Maclaurin's series and simplified their answer fully.

Question 3

(a) The majority of candidates accurately differentiated the product given in the question. Those who then replaced x^2 with x^2+1-1 successfully derived the given reduction formula. Others who integrated $x^2\left(1+x^2\right)^{n-1}$ by parts were unsuccessful.

(b) Most candidates correctly applied the reduction formula from **3(a)** and used I_{-1} as their initial value. Those who tried to use I_{-3} as their initial value were unsuccessful.

Question 4

Almost all knew how to approach this question and completed it to a high standard. There were some inaccuracies when solving linear equations to find the particular integral and some errors when substituting initial conditions. A few candidates gave expressions instead of equations as their answer.

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(a) Most candidates formed a correct expression for the sum of the areas of the rectangles and good responses applied the sum of a geometric series, making clear the first term and common ratio, to accurately derive the given result. Errors included incorrect areas and numbers of terms.

(b) Most candidates correctly adapted their solution to **5(a)** and derived a suitable upper bound. There were some difficulties when simplifying the algebraic expressions which involved different exponents. Final responses were often in a variety of forms.

(c) Good candidates showed clearly that the difference between U_n and L_n is proportional to $\frac{1}{n}$, leading efficiently to the given answer.

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(a) Almost all read the diagonal of $\mathbf{P}$ and stated the eigenvalues.

(b) (i) Almost all maintained accuracy when squaring $\mathbf{P}$.

(ii) Most manipulated the characteristic equation by multiplying through by $\mathbf{P}^{-1}$. Good candidates accurately substituted $\mathbf{P}^2$ when making $\mathbf{P}^{-1}$ the subject.

(c) Most candidates applied $\mathbf{A} = \mathbf{PDP}^{-1}$ using their answer to **6(b)(ii)**. A few candidates took the lengthy approach of finding $\mathbf{P}^{-1}$ again without using the characteristic equation.

Question 7

(a) This was done successfully with candidates either differentiating the logarithmic form of $\tanh^{-1} x$ or differentiating implicitly.

(b) Almost all divided through by x and found the integrating factor. Good responses maintained accuracy and fully simplified the right hand side of the equation after multiplying by the integrating factor. Using integration by parts, most found $\frac{1}{3}x^3 + \frac{1}{3}\left(x^2+16\right)^{\frac{3}{2}} + C$ as the integral of the right hand side and substituted the initial conditions to find the correct value of C.

Question 8

(a) Most candidates stated the formula for the sum of the geometric progression accurately. A few wrote the exponent as n, which hindered their progress in **8(b)**. A few incorrectly stated that the sum was equal to zero.

(b) Almost all knew that de Moivre's theorem related the series to the sum of the geometric progression in **8(a)**. The strongest candidates took the imaginary part, after multiplying the numerator and denominator by the conjugate of z – 1. Use of $z = \frac{1}{2}e^{i\theta}$ or appropriate trigonometric identities then led to the given answer.

(c) Only the very strongest candidates differentiated the equation given in **8(a)** to find $\sum_{m=1}^{n}\left(\frac{1}{2}\right)^m m\cos m\theta$ in terms of n and θ.

(d) The few candidates who had a correct expression for **8(c)** were usually able to apply the given limit successfully.

FURTHER MATHEMATICS

Paper 9231/24
Further Pure Mathematics 2

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.

Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed a thorough understanding of hyperbolic functions and de Moivre's theorem for complex numbers. Most candidates had been well prepared for the examination, but some relied on their calculators too much and were not able to score all marks for showing method and technique. There were many scripts of a high standard.

Comments on specific questions

Question 1

(a) This was done very successfully with the majority of candidates choosing parts appropriately. A few omitted brackets or limits in their working.

(b) This part was generally well done. Most candidates successfully applied the reduction formula to establish the relationship between I_0 and I_2. A few failed to evaluate $\cosh(0) = 1$, which led to an incorrect final answer.

Question 2

(a) The majority of candidates used a hyperbolic identity to reach $\sinh x = \pm\frac{1}{2}$ or $\tanh x = \pm\frac{1}{\sqrt{5}}$.

Those who reduced to $\cosh x = \frac{1}{2}\sqrt{5}$ often missed a solution. Substituting in exponential forms and deriving a quadratic equation in e^{2x} was also commonly seen. Good responses then used a logarithmic form and gave two answers in the form $\ln a$, where a was simplified.

(b) Most candidates drew at least one of the curves correctly. Good responses showed the intersections and symmetry clearly and did not truncate their sketches.

Question 3

(a) After expanding $(\cos\theta + \mathrm{i}\sin\theta)^5$ using the binomial expansion, most candidates grouped together terms contributing to the imaginary part before applying the identity $\cos^2\theta + \sin^2\theta = 1$ to fully justify the given result. Alternatively, a few candidates worked from the right-hand side to the left hand side, applying $2\mathrm{i}\sin\theta = z - z^{-1}$, and were usually successful too.

(b) Almost all rearranged the polynomial equation correctly, linking with the previous part of the question by substituting $x = \sin\theta$. Solving $\sin 5\theta = \frac{1}{2}\sqrt{3}$, most candidates obtained $x = \sin\left(\frac{1}{15}\pi\right)$.

A few candidates gave the values for θ rather than actual solutions, highlighting the need to read the question carefully. Although the four other solutions were often given in different ways, few candidates gave repeated solutions.

Question 4

(a) Most candidates formed a correct expression for the sum of the areas of the rectangles and good responses applied the sum of a geometric series, making clear the first term and common ratio, to accurately derive the given result. Errors included incorrect areas and numbers of terms.

(b) Most candidates correctly adapted their solution to **4(a)** and derived a suitable upper bound. There were some difficulties with exponents when simplifying the algebraic expressions.

(c) Good solutions showed clearly that the difference between U_N and L_N is proportional to $\frac{1}{N}$, leading efficiently to the given answer.

Question 5

(a) Almost all knew how to approach this question and completed it to a high standard. There were some inaccuracies when comparing coefficients to find the particular integral and some problems with notation. A few candidates gave expressions instead of equations as their answer.

(b) Most candidates correctly used their particular integral from **5(a)**. There were also some problems with notation for this part, with a few candidates using an arrow instead of an equals sign. Most could find $R = \sqrt{\frac{2}{13}}$ successfully but sometimes the numerator and denominator of $\tan\phi$ had been reversed.

Question 6

While many candidates successfully solved the differential equation, a significant number did not recognise the importance of signs and constants when determining the integrating factor. Working with $x^2 + 2x - 1$, rather than $1 - 2x - x^2$, changed the entire integration process. Among those who obtained the correct answer, some chose to integrate the expression from first principles rather than using the standard result provided. While not incorrect, this approach was often inefficient and sometimes led to unnecessary algebraic errors. This highlights the importance of being familiar with and effectively using the List of formulae (MF19) during the exam.

Question 7

(a) (i) This was done very well with the majority of candidates showing enough working, particularly when differentiating $\ln\left(\tan\frac{1}{2}t\right)$, to justify the given answer.

(ii) Almost all candidates were able to differentiate y with respect to t.

(iii) The majority of candidates accurately recalled the formula for the length of the curve with correct limits. Most candidates fully simplified $\sqrt{\dot{x}^2 + \dot{y}^2}$ before substituting into the formula, which caused fewer errors and enabled a clear path to the answer.

(b) The attempts to find the second derivative varied in length, with strong candidates showing the required level of algebraic fluency and remembering to divide by $\frac{dx}{dt}$ after differentiating with respect to t. Candidates should be reminded that for the second derivative in parametric form, the application of the chain rule is essential.

Question 8

(a) The majority of candidates found the determinant as a linear expression in terms of a. They then set this not equal to zero to successfully find the set of values of a. Good candidates emphasised that the three planes intersect at a single point.

(b) After subbing in the given value of a, good candidates showed their working clearly when manipulating the equations to arrive at a contradiction. Strong candidates emphasised that the two parallel planes are distinct. A few candidates drew a simple diagram to explain their answer.

(c) Candidates who used the vector product method to find the eigenvectors tended to be most successful, although sign errors were common. Candidates should be encouraged to check that their proposed eigenvector does have the required property by performing matrix multiplication. Almost all showed an awareness of how to find the matrices **P** and **D**. A few did not perform the full number of operations on the eigenvalues of **A** to form **D**.

FURTHER MATHEMATICS

Paper 9231/31
Further Mechanics

Key messages

In all questions, candidates are advised to show complete working, as credit is given for method as well as accuracy. This is particularly the case when a result is given in a question.

A diagram is often an invaluable tool in helping a candidate to make good progress. This is especially the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram as well.

General comments

Candidates are encouraged to draw a suitable diagram or, in the case a diagram is provided, to annotate it, as this helps candidates to understand the problem and model it correctly.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of elementary algebra.

Comments on specific questions

Question 1

Most candidates answered this question well. Successful responses provided equations using the conservation of linear momentum and Newton's experimental law and then combined these, together with the given condition that after the collision the momentum of *A* is three times the momentum of *B*. Some candidates made sign errors or arithmetical errors, and others used the given condition incorrectly.

Question 2

Candidates found this question demanding. The most common error was in finding the radius of the circular motion in the second situation. Many candidates assumed it was the same in both situations, usually calling it r instead of $r\sin\theta_1$ in the first situation and instead of $r\sin\theta_2$ in the second situation.

Question 3

(a) Many candidates obtained the correct expression for v as $v = 2\tan\left(\frac{1}{4}\pi - t\right)$. Some candidates made good attempts but with sign or arithmetical errors. Other candidates did not recognise the integral and incorrectly obtained a logarithmic expression. Stronger responses recognised that having found v in terms of t, the next step is to integrate again to find x in terms of t. This proved to be a challenging step for a significant number of candidates.

(b) This part proved challenging for many candidates. Stronger responses demonstrated an understanding of the requirements of this question. Only the strongest candidates were successful in manipulating the term involving ln(cos(…)) to deduce the required time.

Question 4

Almost all candidates recognised that an energy equation was required. For this situation, successful responses provided a correct energy equation including one kinetic energy term, two elastic potential energy terms and at least one gravitational potential energy term (although the latter may appear as two separate terms). A variety of approaches were seen for determining the length of the string. Successful responses typically introduced an unknown variable, with the most popular choices being the length of the string when the speed was $\sqrt{\frac{7}{5}ag}$ or the extension in the string when P was released. The strongest responses made it clear to the examiner what their unknown variable represented. It is good practice that in any solution, the notation that is introduced is defined. All variations lead to a quadratic equation in the unknown length and a. Amongst weaker responses, a variety of errors in the initial energy equation were seen including omission of at least one of the elastic potential energy terms, errors in the gravitational potential energy term, sign errors or including terms that were not dimensionally correct.

Question 5

(a) Successful responses commonly used the coordinates of the vertices of the triangle and stated that the centre of mass has coordinates at the average of these. Other successful responses set up moments equations to find these coordinates, although such an approach was often less efficient.

(b) This part proved to be challenging for many candidates. Many candidates were successful in setting up a suitable moments equation to find the x-coordinate of the centre of mass of the whole lamina $OABCD$. Successful responses recognised that, for equilibrium, this x-coordinate must be less than or equal to ka, leading to the condition $0 \leqslant h \leqslant \sqrt{3}ak$. A common misconception was that the x-coordinate must be less than ka to ensure equilibrium.

(c) Only the strongest candidates were successful in attempting this part. Successful responses recognised that for the lamina to be on the point of toppling, the upper limit of h in the inequality in **part (b)** needs to be achieved for the x-coordinate. The y-coordinate is most efficiently obtained by recognising that triangle ABC is isosceles for the given value of k and so the lamina is symmetrical.

Question 6

(a) Most candidates successfully used energy conservation between the initial position of P and the lowest point of its motion to find the speed of P immediately before its collision with Q.

(b) Candidates found this part challenging. Many candidates recognised that consideration of the motion of P between the lowest and highest points of its circular path was required. Successful responses attempted the relevant energy equation in combination with use of Newton’s second law of motion at the highest point where the string becomes slack.

(c) Successful responses recognised the requirement to use both the conservation of linear momentum and Newton’s experimental law. Such responses often included a simple diagram, indicating the directions in which the two particles were moving before and after the collision. Responses that did not include a diagram were often prone to sign errors. Some candidates opted for an equation using conservation of kinetic energy. Because the collision is elastic, this is a valid approach, although it is not usually the case in situations involving collisions.

Question 7

(a) Two common methods of solution were seen in this part. The first method was the one suggested in the question, involving differentiation of the equation of the trajectory. The second method was to use *suvat* equations. In both cases, the key insight required is that when the direction of motion of P makes an angle of $45°$ with the downward vertical through A, the tangent of the angle is -1. At the point A, the particle P is on the downward part of its parabolic path. A common error was to omit the minus sign, corresponding to P being on the upward part of its path. The y-coordinates are the same in both cases, but the x-coordinates are different. Candidates who drew a diagram were more likely to avoid such a confusion.

(b) Whilst many candidates obtained the correct answer to this problem, it is possible to obtain the correct answer from an incorrect method if considering P being on the upward part of its path instead of the downward part. Successful responses demonstrated an understanding that the vertical velocity of P is in a downwards direction at point A.

(c) Only the strongest candidates were successful in providing a fully correct solution to this problem. A common error resulted from confusion about the direction in which Newton's experimental law was applied. Candidates often applied the law to a component of the velocity rather than the speed found in **part (b).**

FURTHER MATHEMATICS

Paper 9231/32
Further Mechanics

Key messages

In all questions, candidates are advised to show complete working, as credit is given for method as well as accuracy. This is particularly the case when a result is given in a question.

A diagram is often an invaluable tool in helping a candidate to make good progress. This is especially the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram as well.

General comments

Candidates are encouraged to draw a suitable diagram or, in the case a diagram is provided, to annotate it, as this helps candidates to understand the problem and model it correctly.

Candidates are reminded that, when the answer is given, as in **Questions 4(a)** and **7(a)**, they are expected to show their working in full, even if it involves the use of simple algebra.

Comments on specific questions

Question 1

Most candidates recognised the requirement to resolve the forces at P vertically and apply Newton's second law of motion towards the centre of the circle. As there are two strings attached to P, successful responses recognised that the tensions in each string are not equal. Weaker responses assumed that the tensions in both strings were the same, equivalent to saying that there was one continuous string.

Question 2

(a) Successful responses equated the loss in elastic potential energy in the string to the work done by the friction force. Many candidates instead incorrectly equated the tension in the string to the frictional force.

(b) Stronger responses recognised the need to equate the loss in elastic potential energy, this time involving two terms, to the work done by the friction force. Less successful responses commonly used incorrect distances in the various terms. These errors were less frequent amongst candidates who had drawn a simple diagram illustrating the situation.

Question 3

(a) Successful responses demonstrated accurate working to obtain a correct expression for the distance of the centre of mass above the horizontal surface *BC*, ensuring that any fractions within fractions were simplified when giving a final answer. The most common error was using an incorrect distance for the centre of mass of the quarter circle *AOB* from *BC*. This was usually $a-\frac{4\sqrt{2}}{3\pi}$ instead of $a-\frac{4\sqrt{2}}{3\pi}\cos 45°$. Other candidates used $\frac{4\sqrt{2}}{3\pi}\cos 45°$ or $\frac{4\sqrt{2}}{3\pi}$. A minority of candidates had a moments equation that was dimensionally incorrect.

(b) There were two common approaches to this part. One was to take moments about the point *B*. The other was to equate the result of **part (a)** to *ka*. Only the strongest candidates were able to apply one of these methods successfully. A notable proportion of candidates did not attempt this part.

Question 4

(a) Most candidates attempted this part, but a very common error was to assume that particle *Q* was projected at the same speed as particle *P*. This leads to the false conclusion that α is equal to β, and although the given result can be obtained, it is not a valid method.

(b) This part was often answered correctly, even when **part (a)** was not completed successfully, by using the given result together with various *suvat* equations.

Question 5

(a) Many candidates successfully found the correct expression for *v* in terms of *u* by considering the motion of sphere *A* perpendicular to the line of centres before and after the collision. A minority of candidates used cosine instead of sine in their resolution perpendicular to the line of centres.

(b) Most candidates considered the motion along the line of centres, using conservation of linear momentum and Newton's experimental law. Successful responses solved the two resulting equations to obtain an expression for the coefficient of restitution in terms of *k*. Most errors were either sign or algebraic slips in one or both equations.

(c) Most candidates indicated that the range of possible values for *k* resulted from the fact that the coefficient of restitution is between 0 and 1 (both inclusive). A common error was to assume that the coefficient of restitution could not attain at least one of its limiting values of 0 and 1, so the range of values for *k* was given with either one or two strict inequalities. Weaker responses included errors in the algebraic manipulation of the expression obtained in **part (b)**.

Question 6

Only stronger candidates were successful in providing a fully correct solution to this problem. Successful responses demonstrated a methodical approach making each stage of the working clear. Although the question looked slightly different to the standard questions on this topic, the approach of writing down energy equations and using Newton's second law of motion would have enabled candidates to be awarded most of the marks in this question. Weaker responses made very little progress, often just writing down a single energy equation without a clear strategy to make further progress.

Question 7

(a) Many candidates successfully set up the correct differential equation and solved it by separating the variables, integrating and using the initial condition. Some candidates had an incorrect sign in the differential equation or made algebraic slips in the working. A minority of candidates attempted to solve the problem by using *suvat* equations, even though the acceleration was not constant.

(b) Successful responses recognised that this part requires an expression linking x and v, using acceleration in the form $v\frac{\mathrm{d}v}{\mathrm{d}x}$. The working was typically simpler for those who substituted the given numerical values at an early point in their solution instead of working entirely algebraically. Weaker responses commonly were successful in separating the variables but did not demonstrate a suitable method for integrating the result.

(c) Only the strongest candidates were able to apply the correct formula to find the average speed.

FURTHER MATHEMATICS

Paper 9231/33
Further Mechanics

Key messages

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General comments

Candidates are encouraged to draw a suitable diagram or, in the case a diagram is provided, to annotate it, as this helps candidates to understand the problem and model it correctly.

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Comments on specific questions

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Question 2

Candidates found this question demanding. The most common error was in finding the radius of the circular motion in the second situation. Many candidates assumed it was the same in both situations, usually calling it r instead of $r\sin\theta_1$ in the first situation and instead of $r\sin\theta_2$ in the second situation.

Question 3

(a) Many candidates obtained the correct expression for v as $v = 2\tan\left(\frac{1}{4}\pi - t\right)$. Some candidates made good attempts but with sign or arithmetical errors. Other candidates did not recognise the integral and incorrectly obtained a logarithmic expression. Stronger responses recognised that having found v in terms of t, the next step is to integrate again to find x in terms of t. This proved to be a challenging step for a significant number of candidates.

(b) This part proved challenging for many candidates. Stronger responses demonstrated an understanding of the requirements of this question. Only the strongest candidates were successful in manipulating the term involving ln(cos(…)) to deduce the required time.

Question 4

Almost all candidates recognised that an energy equation was required. For this situation, successful responses provided a correct energy equation including one kinetic energy term, two elastic potential energy terms and at least one gravitational potential energy term (although the latter may appear as two separate terms). A variety of approaches were seen for determining the length of the string. Successful responses typically introduced an unknown variable, with the most popular choices being the length of the string when the speed was $\sqrt{\frac{7}{5}ag}$ or the extension in the string when P was released. The strongest responses made it clear to the examiner what their unknown variable represented. It is good practice that in any solution, the notation that is introduced is defined. All variations lead to a quadratic equation in the unknown length and a. Amongst weaker responses, a variety of errors in the initial energy equation were seen including omission of at least one of the elastic potential energy terms, errors in the gravitational potential energy term, sign errors or including terms that were not dimensionally correct.

Question 5

(a) Successful responses commonly used the coordinates of the vertices of the triangle and stated that the centre of mass has coordinates at the average of these. Other successful responses set up moments equations to find these coordinates, although such an approach was often less efficient.

(b) This part proved to be challenging for many candidates. Many candidates were successful in setting up a suitable moments equation to find the x-coordinate of the centre of mass of the whole lamina $OABCD$. Successful responses recognised that, for equilibrium, this x-coordinate must be less than or equal to ka, leading to the condition $0 \leqslant h \leqslant \sqrt{3}ak$ **.** A common misconception was that the x-coordinate must be less than ka to ensure equilibrium.

(c) Only the strongest candidates were successful in attempting this part. Successful responses recognised that for the lamina to be on the point of toppling, the upper limit of h in the inequality in **part (b)** needs to be achieved for the x-coordinate. The y-coordinate is most efficiently obtained by recognising that triangle ABC is isosceles for the given value of k and so the lamina is symmetrical.

Question 6

(a) Most candidates successfully used energy conservation between the initial position of P and the lowest point of its motion to find the speed of P immediately before its collision with Q.

(b) Many candidates recognised that consideration of the motion of P between the lowest and highest points of its circular path was required. Successful responses attempted the relevant energy equation in combination with use of Newton's second law of motion at the highest point where the string becomes slack.

(c) Successful responses recognised the requirement to use both the conservation of linear momentum and Newton's experimental law. Such responses often included a simple diagram, indicating the directions in which the two particles were moving before and after the collision. Responses that did not include a diagram were often prone to sign errors. Some candidates opted for an equation using conservation of kinetic energy. Because the collision is elastic, this is a valid approach, although it is not usually the case in situations involving collisions.

Question 7

(a) Two common methods of solution were seen in this part. The first method was the one suggested in the question, involving differentiation of the equation of the trajectory. The second method was to use *suvat* equations. In both cases, the key insight required is that when the direction of motion of P makes an angle of $45°$ with the downward vertical through A, the tangent of the angle is –1. At the point A, the particle P is on the downward part of its parabolic path. A common error was to omit the minus sign, corresponding to P being on the upward part of its path. The y-coordinates are the same in both cases, but the x-coordinates are different. Candidates who drew a diagram were more likely to avoid such a confusion.

(b) Whilst many candidates obtained the correct answer to this problem, it is possible to obtain the correct answer from an incorrect method if considering P being on the upward part of its path instead of the downward part. Successful responses demonstrated an understanding that the vertical velocity of P is in a downwards direction at point A.

(c) Only the strongest candidates were successful in providing a fully correct solution to this problem. A common error resulted from confusion about the direction in which Newton's experimental law was applied. Candidates often applied the law to a component of the velocity rather than the speed found in **part (b).**

FURTHER MATHEMATICS

Paper 9231/34
Further Mechanics

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved.

If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagram as well.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the examiner is in no doubt that the offered solution is clear and complete. In all questions, candidates are advised to show all their working, as credit is given for method as well as accuracy.

General comments

Candidates are encouraged to draw a suitable diagram or, in the case a diagram is provided, to annotate it, as this helps candidates to understand the problem and model it correctly. For example, this supported many candidates with the problem-solving element of **Question 2 part (b)**.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments and conservation of energy equations. When applying Newton's second law in contexts such as setting up a differential equation or questions involving collisions, candidates must ensure they explicitly mention the mass(es) involved.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of elementary algebra.

Candidates should also be reminded to correctly set their calculator to degrees or radians, according to what is required in the question. In particular, questions involving calculus will require working in radians. This impacted some candidates, for example, in answering **Question 5 part (b)**.

Comments on specific questions

Question 1

This question was well attempted by most candidates. Successful responses showed full detail in the working and demonstrated a high level of fluency in algebraic manipulation. Successful candidates ensured that all stages of their working were presented clearly and logically. For example, successful solutions clearly demonstrated how the length of BR is obtained.

Less successful responses considered the mass of particle B to be m instead of $0.5m$. A common misunderstanding seen was $BR = 12a$. Successful candidates usually did not make this misunderstanding by drawing or annotating the diagram.

Question 2

(a) Candidates performed well in this question, with many successfully substituting the required values into the trajectory equation from the list of formulae provided in MF19. As the result is given in the question, it is essential that candidates show all their working to clearly demonstrate how they reach the equation given in the question.

(b) Successful responses recognised the requirement to substitute $x = 8$ and $y = 12$ into the equation from **part (a)** to obtain $u^2 = 400$ (or, equivalently $u = 20$). Many candidates were successful in this first stage of the question.

Success in the latter stage of this question depended upon recognising that there are two conditions to consider, namely $y \geqslant 0$ and $y \leqslant 7$. Candidates who drew a diagram were often successful in realising the need to consider both conditions. Less successful responses only considered one of the conditions, most commonly $y \leqslant 7$ only.

Question 3

(a) Many candidates were successful in this part question. Successful responses demonstrated clear and well-organised working, with strong levels of algebraic manipulation. Less successful responses typically made algebraic slips in the working, with the correct given answer for $\bar{x}$ commonly quoted following incorrect working.

(b) Successful responses recognised the need to use the condition $\bar{x} \geqslant a$. Many responses recognised that this leads to the solution $0 < h \leqslant a$, with only the strongest responses recognising that $h = 2a$ is also part of the solution.

Question 4

(a) Many candidates found this part question challenging. Stronger responses usually included a diagram or labelled the diagram provided. Less successful responses often included only one kinetic energy term in the equation for the conservation of mechanical energy. Candidates who were successful in this problem recognised that the condition at the top of the shell for the particle to move in complete vertical circles is $R \geqslant 0$, not simply $R = 0$.

(b) Successful candidates recognised that the greatest possible value of the normal reaction will occur at the bottom of the shell. Less successful responses provided energy equations that were not dimensionally correct.

Question 5

(a) Many candidates successfully set up the correct differential equation and attempted to solve it. Successful candidates included the mass when setting up an equation using Newton's second law, going on to cancel the mass in subsequent working. Less successful responses used opposite signs for the gravity and the resistive force or omitted the gravity term in the differential equation.

(b) Successful candidates set up the appropriate differential equation, recognising the need to use $a = \frac{dv}{dt}$. Many candidates recognised that the question requires the use of radian measure throughout, due to the calculus involved. In setting up the differential equation, omissions and errors were similar to those in **part (a)**, including omitting the mass or considering opposite signs for the gravity and resistive forces.

Question 6

(a) A variety of successful methods were seen in answering this problem. The most common and successful approach seen was to express Hooke's law in both strings in terms of BP. Other popular approaches involved setting x equal to the extension in string AP, or to the extension in string BP, or to consider two unknowns, one for the extension in string AP, the other one for the extension in string BP. Successful responses often included a suitable diagram. Weaker responses commonly attempted to use energy equations without recognising the connection between the extensions in the strings *AP* and *BP*.

(b) Successful responses set up a suitable energy equation at the point of release and the point where *P* comes to instantaneous rest. Less successful responses commonly considered the motion in separate stages, with such responses more prone to algebraic slips or errors in the energy terms.

Question 7

Successful responses to this challenging final question recognised the requirement to combine the equations involving the speed of the particle before and after the collision, the distance travelled, and Newton's experimental law. Some elegant responses were provided that successfully combined all the equations in one step, allowing for an efficient approach to solving the problem. Examples of such approaches are:
$\frac{4}{5}eukT = \frac{4}{5}uT$, $eu\sin\alpha = v\sin\theta = \dfrac{d}{kT} = \dfrac{u\sin\alpha}{k}$, $e = \dfrac{v}{u\sin\alpha} = \dfrac{u\sin\alpha}{k} \times \dfrac{1}{u\sin\alpha} = \dfrac{1}{k}$ and
$\dfrac{d}{u\sin\alpha} : \dfrac{d}{eu\sin\alpha} = 1 : k$.

FURTHER MATHEMATICS

Paper 9231/41
Further Probability & Statistics

Key messages

- In all questions, sufficient method must be shown to justify answers.
- When performing a hypothesis test, candidates should clearly state the result of the test in terms of accepting or rejecting the null hypothesis before giving their conclusion in context.
- Care must be taken with the language used when interpreting the result of any test.
- Candidates need to work to the required level of accuracy of three significant figures. To maintain accuracy of 3 significant figures in the final answer, all intermediate working should be to at least 4 significant figures.

General comments

Most candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

Care must be taken with the language used when interpreting the result of a hypothesis test. In general, a hypothesis test is not a proof, and it is not appropriate to use definite statements for the conclusion. As such, concluding statements should include some degree of uncertainty, for example, ‘there is sufficient evidence to support the claim …’, rather than ‘the test proves that …’.

Comments on specific questions

Question 1

(a) Most candidates applied a suitable method for finding the confidence interval. Weaker responses used an incorrect *t*-value, typically using 1.833 or 1.860 in place of 2.262. Successful responses ensured that the final answer was expressed as an interval using either interval notation or inequalities. Only the strongest candidates provided at least one fully correct assumption. Many responses referred to normality without explicitly referring to the underlying population or random variable. Another common valid response included reference to the observations/data being assumed to be independent.

(b) Successful responses clearly explained why the assumption made in **part (a)** may not be appropriate with some reference to the context or data. The strongest responses made reference to all of the observations being less than $90°$, since the angle is acute, or the presence of an extreme value/outlier, $77°$. These both cast some doubt as to whether the underlying population can be assumed to be normal. Some candidates suggested reasons why the observations may not be independent, for example if the students were working together when estimating the angle, which was an acceptable response.

Question 2

(a) The majority of candidates were able to correctly calculate the required values. Where errors occurred in the working, these were typically numerical slips. A minority of candidates obtained the correct probability but did not multiply this by 100 to obtain the correct expected frequency.

(b) Most candidates were successful in calculating the test statistic and demonstrated their method clearly. A small minority of candidates combined columns despite all the expected frequencies being at least 5. Most candidates made an appropriate comparison of their test statistic with the correct critical value leading to the correct test result. Weaker responses did not specify the hypotheses fully, often omitting the mean rate $\lambda = 2.8$. Other responses gave conclusions that were too definitive or inappropriately expressed in terms of the null hypothesis.

Question 3

Many candidates were successful in solving this question. Some candidates confused the concepts of standard deviation and variance, both with the calculation, where 0.6 in place of 0.6^2 was sometimes seen, and also with the final answer where 0.302 in place of $\sqrt{0.302} = 0.550$ was sometimes seen.

Question 4

Candidates found this question challenging. When finding an expression for the test statistic, only the strongest candidates correctly applied the required continuity correction. Weaker responses considered the upper tail of the distribution instead of the required lower tail.

Question 5

(a) Almost all candidates correctly obtained the given value for *k* with sufficient working shown.

(b) Successful candidates identified the correct interval for the median and provided a suitable equation by considering the area above or below the median. A minority of candidates struggled to make any progress with this question.

(c) This part was generally well attempted with many fully correct responses for the probability density function seen. Weaker responses provided probability density functions that did not cover all real values, typically omitting 0 otherwise (or equivalent).

Question 6

(a) Most candidates were successful in forming the hypotheses. Weaker responses did not explicitly refer to the population in their hypotheses, either by omitting the word population or by including the symbol for the sample mean as part of the hypotheses. Successful responses recognised the requirement to calculate the signed differences. Although many appropriate conclusions in context were seen, it was not uncommon to see statements such as ‘this shows that the coach is correct’ which lacks the level of uncertainty required when making inferences based on the result of a hypothesis test.

(b) Only the strongest candidates were able to express clearly that for the test in **part (a)** to be valid it must be assumed that the underlying population of differences is normally distributed.

(c) Only the strongest candidates correctly identified that a paired non-parametric test is appropriate. Weaker responses named a non-parametric test without reference to a paired being required. A minority of candidates incorrectly commented that the further research suggests that a paired test is no longer be appropriate.

Question 7

(a) Almost all candidates were able to write down the correct probability generating function of X and use its derivative to find the expectation of X.

(b) Almost all candidates knew how to use the probability generating function of X to find its variance. Successful responses ensured that sufficient working was shown when deriving the expression that is given in the question.

(c) Most candidates recognised that the probability generating function of Y is given by the probability generating function of X raised to the power 10. A minority of candidates worked instead with ten times the probability generating function of X. Attempts to work with $\mathrm{G}_Y(t) = \mathrm{G}_X(t^{10})$ were also seen.

(d) Successful candidates recognised that, for the case when $b = 0$, the random variable X reduces to a binomial distribution. Whilst many candidates made this connection, the most common incomplete response was stating 'binomial' without specifying the parameters for this distribution.

FURTHER MATHEMATICS

Paper 9231/42
Further Probability & Statistics

Key messages

- In all questions, sufficient method must be shown to justify answers.
- When performing a hypothesis test, candidates should clearly state the result of the test in terms of accepting or rejecting the null hypothesis before giving their conclusion in context.
- Care must be taken with the language used when interpreting the result of any test.
- Candidates need to work to the required level of accuracy of three significant figures. To maintain accuracy of 3 significant figures in the final answer, all intermediate working should be to at least 4 significant figures.

General comments

Most candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

Care must be taken with the language used when interpreting the result of a hypothesis test. In general, a hypothesis test is not a proof, and it is not appropriate to use definite statements for the conclusion. As such, concluding statements should include some degree of uncertainty, for example, ‘there is sufficient evidence to support the claim that …’, rather than ‘the test proves that …’.

Comments on specific questions

Question 1

(a) Only the strongest responses provided explanations that referred to the given data. Weaker responses provided answers that were too vague or general, without reference to the given data.

(b) Many candidates were successful in performing the sign test. The strongest responses provided a clear comparison of the p-value with 0.05 (or offered a valid equivalent comparison) alongside a clear statement that the null hypothesis should not be rejected, followed by the conclusion in context. Weaker responses included hypotheses that were not fully correct, for example often just stating ‘median’ rather than ‘population median’, and conclusions that did not have the appropriate level of uncertainty in the phrases used.

Question 2

(a) Most candidates correctly obtained the confidence interval for μ. Weaker responses included an incorrect t value or used a z value instead of a t value.

(b) Candidates found this part of the question more challenging. Many candidates attempted to set up an inequality, but this was not always of the correct form. Weaker responses often contained a sign error in the critical value.

Question 3

(a) This was generally well answered with most candidates demonstrating a correct method to find both *m* and *n*. A minority of candidates did not keep to the required level of accuracy. Weaker responses incorrectly considered $P(X = 5)$ when calculating the value of *n*.

(b) Successful responses provided hypotheses that fully specified the distribution, including the parameter of 0.7, as well as linking this distribution to the context (for example, statements such as '…is a satisfactory model for the number of breakdowns per day' or '…is a satisfactory model for the data'). Weaker responses only referred to a Poisson distribution in their hypotheses, rather than Poisson with mean 0.7, whilst other responses did not link this fully to the scenario (for example, simply stating '… is a satisfactory model'). Most candidates recognised the need for combining columns and provided correct calculations for the test statistic. Successful responses provided a valid comparison of the test statistic with the critical value along with a statement to accept H_0 (or equivalent), followed by a conclusion that is in context and non-definite. Weaker responses gave conclusions that were too definitive or inappropriately expressed in terms of the null hypothesis.

Question 4

(a) Many candidates successfully found the correct values for *a* and *b* by recognising both $F(1) = 0$ and $F(6) = 1$.

(b) This part was well attempted with many fully correct responses for the probability density function seen. Weaker responses provided probability density functions that did not cover all real values, typically omitting 0 otherwise (or equivalent).

(c) Most candidates were successful in solving this part of the question. Where errors occurred in the working, these were typically numerical slips. Some candidates inefficiently calculated $E(X)$ despite the value being given in the question.

(d) Many candidates successfully calculated the 10th and 90th percentiles for *X*.

Question 5

Most candidates were successful in performing the required *t*-test. Weaker responses provided conclusions that did not have the required level of uncertainty in the language used, included incorrect comparisons, or made errors in the method when calculating *t*.

Question 6

(a) Many candidates were successful in finding $E(X)$ and $Var(X)$. Weaker responses typically made slips with signs or powers when simplifying the expressions for the first and second derivatives of the probability generating function.

(b) Successful responses recognised the need to perform a binomial expansion, from which the correct probability could be determined. Weaker responses often struggled to identify a suitable approach to make progress with this question.

FURTHER MATHEMATICS

Paper 9231/43
Further Probability & Statistics

Key messages

- In all questions, sufficient method must be shown to justify answers.
- When performing a hypothesis test, candidates should clearly state the result of the test in terms of accepting or rejecting the null hypothesis before giving their conclusion in context.
- Care must be taken with the language used when interpreting the result of any test.
- Candidates need to work to the required level of accuracy of three significant figures. To maintain accuracy of 3 significant figures in the final answer, all intermediate working should be to at least 4 significant figures.

General comments

Most candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

Care must be taken with the language used when interpreting the result of a hypothesis test. In general, a hypothesis test is not a proof, and it is not appropriate to use definite statements for the conclusion. As such, concluding statements should include some degree of uncertainty, for example, ‘there is sufficient evidence to support the claim …’, rather than ‘the test proves that …’.

Comments on specific questions

Question 1

(a) Most candidates applied a suitable method for finding the confidence interval. Weaker responses used an incorrect t-value, typically using 1.833 or 1.860 in place of 2.262. Successful responses ensured that the final answer was expressed as an interval using either interval notation or inequalities. Only the strongest candidates provided at least one fully correct assumption. Many responses referred to normality without explicitly referring to the underlying population or random variable. Another common valid response included reference to the observations/data being assumed to be independent.

(b) Successful responses clearly explained why the assumption made in **part (a)** may not be appropriate with some reference to the context or data. The strongest responses made reference to all of the observations being less than $90°$, since the angle is acute, or the presence of an extreme value/outlier, $77°$. These both cast some doubt as to whether the underlying population can be assumed to be normal. Some candidates suggested reasons why the observations may not be independent, for example if the students were working together when estimating the angle, which was an acceptable response.

Question 2

(a) The majority of candidates were able to correctly calculate the required values. Where errors occurred in the working, these were typically numerical slips. A minority of candidates obtained the correct probability but did not multiply this by 100 to obtain the correct expected frequency.

(b) Most candidates were successful in calculating the test statistic and demonstrated their method clearly. A small minority of candidates combined columns despite all the expected frequencies being at least 5. Most candidates made an appropriate comparison of their test statistic with the correct critical value leading to the correct test result. Weaker responses did not specify the hypotheses fully, often omitting the mean rate $\lambda = 2.8$. Other responses gave conclusions that were too definitive or inappropriately expressed in terms of the null hypothesis.

Question 3

Many candidates were successful in solving this question. Some candidates confused the concepts of standard deviation and variance, both with the calculation, where 0.6 in place of 0.6^2 was sometimes seen, and also with the final answer where 0.302 in place of $\sqrt{0.302} = 0.550$ was sometimes seen.

Question 4

Candidates found this question challenging. When finding an expression for the test statistic, only the strongest candidates correctly applied the required continuity correction. Weaker responses considered the upper tail of the distribution instead of the required lower tail.

Question 5

(a) Almost all candidates correctly obtained the given value for *k* with sufficient working shown.

(b) Successful candidates identified the correct interval for the median and provided a suitable equation by considering the area above or below the median. A minority of candidates struggled to make any progress with this question.

(c) This part was generally well attempted with many fully correct responses for the probability density function seen. Weaker responses provided probability density functions that did not cover all real values, typically omitting 0 otherwise (or equivalent).

Question 6

(a) Most candidates were successful in forming the hypotheses. Weaker responses did not explicitly refer to the population in their hypotheses, either by omitting the word population or by including the symbol for the sample mean as part of the hypotheses. Successful responses recognised the requirement to calculate the signed differences. Although many appropriate conclusions in context were seen, it was not uncommon to see statements such as ‘this shows that the coach is correct’ which lacks the level of uncertainty required when making inferences based on the result of a hypothesis test.

(b) Only the strongest candidates were able to express clearly that for the test in **part (a)** to be valid it must be assumed that the underlying population of differences is normally distributed.

(c) Only the strongest candidates correctly identified that a paired non-parametric test is appropriate. Weaker responses named a non-parametric test without reference to a paired being required. A minority of candidates incorrectly commented that the further research suggests that a paired test is no longer be appropriate.

Question 7

(a) Almost all candidates were able to write down the correct probability generating function of *X* and use its derivative to find the expectation of *X*.

(b) Almost all candidates knew how to use the probability generating function of *X* to find its variance. Successful responses ensured that sufficient working was shown when deriving the expression that is given in the question.

(c) Most candidates understood that the probability generating function of *Y* is given by the probability generating function of *X* raised to the power 10. A minority of candidates worked instead with ten times the probability generating function of *X*. Attempts to work with $\mathrm{G}_Y(t) = \mathrm{G}_X(t^{10})$ were also seen.

(d) Successful candidates recognised that, for the case when $b = 0$, the random variable *X* reduces to a binomial distribution. Whilst many candidates made this connection, the most common incomplete response was stating 'binomial' without specifying the parameters for this distribution.

FURTHER MATHEMATICS

Paper 9231/44
Further Probability & Statistics

Key messages

- In all questions, sufficient method must be shown to justify answers.
- When performing a hypothesis test, candidates should clearly state the result of the test in terms of accepting or rejecting the null hypothesis before giving their conclusion in context.
- Care must be taken with the language used when interpreting the result of any test.
- Candidates need to work to the required level of accuracy of three significant figures. To maintain accuracy of 3 significant figures in the final answer, all intermediate working should be to at least 4 significant figures.

General comments

Most candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

Care must be taken with the language used when interpreting the result of a hypothesis test. In general, a hypothesis test is not a proof, and it is not appropriate to use definite statements for the conclusion. As such, concluding statements should include some degree of uncertainty, for example, ‘there is sufficient evidence to support the claim that …’, rather than ‘the test proves that …’.

Comments on specific questions

Question 1

(a) Most candidates applied a suitable method for finding the confidence interval. Weaker responses used an incorrect t-value, typically using 1.356 in place of 1.782. Successful responses ensured that the final answer was expressed as an interval using either interval notation or inequalities. Some candidates gave bounds that were not correct to 3 significant figures, commonly because of premature rounding (for example, by using 3 significant figures for the sample mean).

(b) Successful candidates provided a fully correct reason that commented on normality in relation to the underlying population or random variable.

Question 2

(a) Almost all candidates attempted to integrate the given probability density function. Weaker responses did not consider a constant or used an incorrect method to find it. Incomplete responses provided a CDF that did not cover all real values.

(b) Almost all candidates identified the correct interval for the median to form and solve a valid equation. Weaker responses gave a solution that was not exact.

(c) Most candidates were successful in finding the CDF for Y. Weaker responses substituted y^2 directly into the PDF of X and/or did not specify the domain correctly in terms of y.

(d) A variety of approaches were seen for this part. Some candidates opted to evaluate the median of Y, but weaker responses did not explicitly show that this was less than the median of X. Successful responses often gave arguments based on squaring a positive number less than 1 or substituting the median of X into the CDF of Y.

Question 3

(a) Most candidates were successful in forming the hypotheses. Weaker responses did not explicitly refer to the population in their hypotheses, either by omitting the word population or by including the symbol for the sample mean as part of the hypotheses. Successful responses recognised the requirement to calculate the signed differences. Although many appropriate conclusions in context were seen, it was not uncommon to see conclusions suggesting that there was sufficient evidence to support the manufacturer's claim after having failed to reject the null hypothesis.

(b) Many candidates recognised that normality is a key element of the required assumption. Only the strongest responses were fully complete in specifying that, in this case, it is the population of differences which must be normally distributed.

(c) Many successful responses were seen to this part question, with most candidates recognising that the difference remains the same for pair 4.

Question 4

(a) Almost all candidates correctly wrote down the required probability generating function.

(b) Most candidates recognised that the probability generating function of *Y* is given by the probability generating function of *X* raised to the power 4. A minority of candidates worked instead with four times the probability generating function of *X*. Attempts to work with $\mathrm{G}_Y(t) = \mathrm{G}_X(4)$ were also seen.

(c) Most candidates recognised how to obtain the required probability from the probability generating function. Where errors occurred in the working, these were typically algebraic slips.

(d) Most candidates successfully calculated the variance of *Y*.

Question 5

Many candidates provided fully correct responses to this question. Weaker responses provided an incorrect test result following the correct test statistic and critical value being seen. Some candidates stated the hypotheses in the opposite order, whilst others provided incorrect conclusions in context following correct work throughout the rest of the test.

Question 6

(a) Successful candidates were fluent in using a suitable normal approximation as part of the test. Weaker responses were not explicit in referring to the population when forming hypotheses and/or did not apply a correct continuity correction when calculating the test statistic.

(b) Only the strongest candidates provided a valid response to this part. Many candidates provided an assumption about the underlying population being normal, which is not a necessary assumption for a non-parametric test.