Cambridge Assessment
International Education

Example Responses – Paper 2

Cambridge International AS & A Level Further Mathematics 9231

For examination from 2025

Cambridge Pathway

Contents

Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Further Mathematics 9321.

This booklet contains responses to all questions from June 2025 Paper 24, which have been written by a Cambridge examiner. Responses are accompanied by a brief commentary highlighting common errors and misconceptions where they are relevant.

The question papers and mark schemes are available to download from the School Support Hub

9321 June 2025 Question Paper 24

9321 June 2025 Mark Scheme 24

Past exam resources and other teaching and learning resources are available from the School Support Hub

Question 1

1 **(a)** Find the values of k for which the system of equations

$$\begin{aligned} x + 2y + 3z &= 1, \\ kx + 5y + 6z &= 2, \\ 7x + 2ky + 9z &= 3, \end{aligned}$$

does not have a unique solution. [3]

$$\begin{vmatrix} 1 & 2 & 3 \\ k & 5 & 6 \\ 7 & 2k & 9 \end{vmatrix} = 45 - 12k - 2(9k - 42) + 3(2k^2 - 35)$$

$$6k^2 - 30k + 24 = 0 \Rightarrow k = 1, 4$$

Examiner comment

The common errors included incorrect algebra when simplifying to arrive at a three term quadratic; or, stating that $k \neq 14$.

(b) Given that $k = 1$, show that the system of equations in part **(a)** is consistent. Interpret this situation geometrically. [3]

$$\begin{aligned} x + 2y + 3z &= 1, \\ x + 5y + 6z &= 2, \\ 7x + 2y + 9z &= 3, \end{aligned} \Rightarrow \begin{aligned} 3y + 3z &= 1 \\ 3x + 3z &= 1 \end{aligned}$$

three planes meeting in a common line

Examiner comment

- Most candidates were able to correctly state that the three planes meet a common line or formed a sheaf.
- The main errors were from incorrect or unclear working when trying to use the three equations to form two simultaneous equations.
- Successful responses labelled the equations, used all three equations and clearly stated the operations.
- Many candidates reduced to row echelon form with clear row operations.

Question 2

2 Find the exact value of $\int_1^{\frac{5}{2}} \frac{1}{\sqrt{x^2-2x+5}}\,dx$, giving your answer in logarithmic form. [6]

$$x^2 - 2x + 5 = 4 + (x-1)^2$$

$$\int_1^{\frac{5}{2}} \frac{1}{\sqrt{4+(x-1)^2}}\,dx = \left[\sinh^{-1}\left(\frac{x-1}{2}\right)\right]_1^{\frac{5}{2}}$$

$$= \sinh^{-1}\left(\tfrac{3}{4}\right) - \sinh^{-1}(0)$$

$$= \ln\left(\tfrac{3}{4} + \sqrt{\tfrac{9}{16}+1}\right) = \ln 2$$

Examiner comment

- Most candidates completed this question successfully, making their use of the logarithmic of $\sinh^{-1}$ explicit.
- Some candidates completed the square incorrectly or applied the wrong formula when integrating.

Question 3

3 Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 13e^{3x}$$

given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. [10]

$m^2 + 4m + 5 = 0 \Rightarrow m = -2 \pm i$

$y = e^{-2x}\left(A\cos x + B\sin x\right)$

$y = ke^{3x} \Rightarrow y' = 3ke^{3x} \Rightarrow y'' = 9ke^{3x}$

$9k + 12k + 5k = 13 \Rightarrow k = \frac{1}{2}$

$y = e^{-2x}\left(A\cos x + B\sin x\right) + \frac{1}{2}e^{3x}$

$A + k = 1 \Rightarrow A = 1 - k = \frac{1}{2}$

$y' = e^{-2x}\left(-A\sin x + B\cos x\right) - 2e^{-2x}\left(A\cos x + B\sin x\right) + \frac{3}{2}e^{3x}$

$B - 2A + \frac{3}{2} = 0 \Rightarrow B = -\frac{1}{2}$

$y = \frac{1}{2}e^{-2x}\left(\cos x - \sin x\right) + \frac{1}{2}e^{3x}$

Examiner comment

- Most candidates knew how to approach this question and completed it to a high standard.
- There were some inaccuracies when candidates compared coefficients to find the particular solution, and some problems with notation.
- Some candidates gave expressions instead of equations as their answer.

Question 4

4 A curve has parametric equations

$$x = t^3 - t^2 + t - 1 \qquad \text{and} \qquad y = te^t.$$

(a) Show that 1 is the only real value of t for which $x = 0$. [1]

$$x = 0 \Rightarrow (t-1)(t^2+1) = 0$$

Examiner comment

- This question was answered well with most candidates factorising the cubic as $(t-1)(t^2+1) = 0$ to show that there was only one real root.
- Many candidates alternatively found the complex roots $\pm$i.

(b) Show that $\dfrac{dy}{dx} = \dfrac{(t+1)e^t}{3t^2-2t+1}$. [3]

$$\frac{dx}{dt} = 3t^2 - 2t + 1 \qquad \frac{dy}{dt} = (t+1)e^t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{(t+1)e^t}{3t^2-2t+1}$$

Examiner comment

- Most candidates found the first derivative correctly using parametric differentiation.
- Sign errors occasionally occurred when deriving the given answer.

(c) Find the Maclaurin's series for y up to and including the term in x^2. [6]

$$\frac{d}{dt}\left(\frac{(t+1)e^t}{3t^2-2t+1}\right) = \frac{\left(3t^2-2t+1\right)\left((t+1)e^t + e^t\right) - (t+1)e^t\left(6t-2\right)}{\left(3t^2-2t+1\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(3t^2-2t+1\right)(t+2)e^t - (t+1)\left(6t-2\right)e^t}{\left(3t^2-2t+1\right)^3}$$

$$y(0) = e, \text{ and } y(0) = e, \quad y'(0) = e, \quad y''(0) = -\tfrac{1}{4}e$$

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 = e + ex - \tfrac{1}{8}ex^2$$

Examiner comment

- The attempts to find the second derivative varied in length. Successful candidates showed the required level of algebraic fluency and remembered to divide by $\frac{dx}{dt}$ after differentiating with respect to t.
- Candidates should be reminded that for the second derivative in parametric form, the application of the chain rule is essential.
- For the Maclaurin's series, a few candidates incorrectly substituted $t = 0$.

Question 5

5 **(a)** Use de Moivre's theorem to show that

$$\sin 7\theta = -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta. \qquad [5]$$

$\sin 7\theta = \operatorname{Im}\left(\left(c + is\right)^7\right)$

$= -s^7 + 21c^2s^5 - 35c^4s^3 + 7c^6s$

$= -s^7 + 21\left(1 - s^2\right)s^5 - 35\left(1 - s^2\right)^2 s^3 + 7\left(1 - s^2\right)^3 s$

$\left(1 - s^2\right)^3 = -s^6 + 3s^4 - 3s^2 + 1$

Examiner comment

Some candidates made errors in their algebraic manipulation or in the expansion of higher power terms when converting to an expression in terms of $sin\theta$ to reach the given answer.

(b) Hence find all roots of the equation

$$64x^6 - 112x^4 + 56x^2 - 7 = 0$$

in the form $\sin q\pi$, where q is a rational number. [3]

$x = \sin\theta \quad \sin\theta \neq 0 \quad \sin 7\theta = 0$

$x = \sin\left(\pm\frac{1}{7}\pi\right), \quad \sin\left(\pm\frac{2}{7}\pi\right), \quad \sin\left(\pm\frac{3}{7}\pi\right)$

Examiner comment

- Most candidates rearranged the polynomial equation correctly, linking with the previous part of the question by substituting $x = \sin\theta$.
- Successful candidates excluded $\sin\theta = 0$ from their solutions.
- When solving $\sin 7\theta = 0$, most candidates obtained $x = \sin\left(\frac{1}{7}\pi\right)$.
- Some candidates gave the values for θ rather than actual solutions, which highlighted the need for candidates to read the question carefully.
- The five other solutions were often given in different ways, and some candidates gave repeated solutions.

Question 6

6 Find the solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 2x^2\tan^{-1}x$$

for which $y = \frac{1}{2}\pi$ when $x = 1$. Give your answer in the form $y = \mathrm{f}(x)$. [9]

$$\frac{dy}{dx} - \frac{1}{x}y = 2x\tan^{-1}x$$

$$e^{-\int x^{-1}\,dx} = x^{-1}$$

$$\frac{d}{dx}\left(yx^{-1}\right) = 2\tan^{-1}x$$

$$yx^{-1} = 2x\tan^{-1}x - 2\int\frac{x}{1+x^2}\,dx = 2x\tan^{-1}x - \ln\left(1+x^2\right) + C$$

$$\tfrac{1}{2}\pi = 2\left(\tfrac{1}{4}\pi\right) - \ln 2 + C \Rightarrow C = \ln 2$$

$$y = 2x^2\tan^{-1}x - x\ln\left(1+x^2\right) + x\ln 2$$

Examiner comment

- While many candidates successfully solved the differential equation, a significant number failed to recognise the importance of the negative sign when determining the integrating factor. This seemingly small oversight often led to an incorrect integrating factor, which in turn changed the entire integration process.
- Candidates who obtained the correct integrating factor usually also integrated $\tan^{-1}x$ by parts to arrive at the correct general solution.
- After substituting the initial conditions and dividing through by the integrating factor, there was some poor notation where $x\ln 2$ was written as $\ln 2x$.

Question 7

7 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 7 & 11 \\ 0 & 2 & 5 \\ 0 & 0 & -3 \end{pmatrix}.$$

(a) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^6 = \mathbf{PDP}^{-1}$. [7]

eigenvalues of A are 1, 2 and −3.

$$\lambda = 1: \quad \begin{vmatrix} i & j & k \\ 0 & 1 & 5 \\ 0 & 0 & -4 \end{vmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \lambda = 2: \quad \begin{vmatrix} i & j & k \\ -1 & 7 & 11 \\ 0 & 0 & 5 \end{vmatrix} = \begin{pmatrix} 35 \\ 5 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = -3: \quad \begin{vmatrix} i & j & k \\ 4 & 7 & 11 \\ 0 & 5 & 5 \end{vmatrix} = \begin{pmatrix} -20 \\ -20 \\ 20 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\text{thus } \mathbf{P} = \begin{pmatrix} 1 & 7 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{thus } \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 729 \end{pmatrix}$$

Examiner comment

- Candidates who used the vector product method to find the eigenvectors tended to be most successful, although sign errors were common.
- Candidates should be encouraged to check that their proposed eigenvector does have the required property by performing matrix multiplication.
- Most candidates showed an awareness of how to find the matrices **P** and **D**.
- A few candidates did not take the 6th power of the eigenvalues of **A** or used a zero vector in their final answer.

(b) Use the characteristic equation of $\mathbf{A}$ to show that

$$\mathbf{A}^6 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I},$$

where a, b and c are integers to be determined. [4]

$$(\lambda - 1)(\lambda - 2)(\lambda + 3) = \lambda^3 - 7\lambda + 6 = 0$$

$$\mathbf{A}^3 = 7\mathbf{A} - 6\mathbf{I}$$

$$\mathbf{A}^6 = (7\mathbf{A} - 6\mathbf{I})(7\mathbf{A} - 6\mathbf{I}) = 49\mathbf{A}^2 - 84\mathbf{A} + 36\mathbf{I}$$

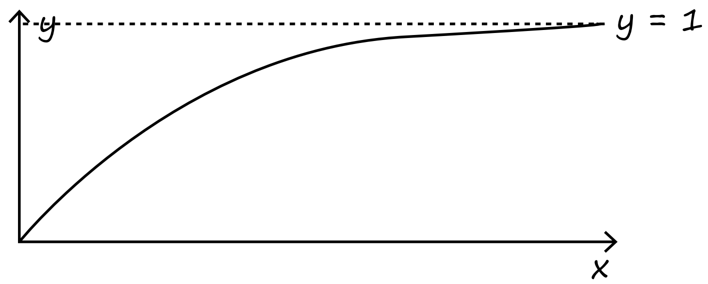
Examiner comment

- Most candidates had the correct characteristic equation and arrived at the answer by making $\mathbf{A}^3$ the subject and squaring both sides.
- A few candidates used lengthier methods, multiplying through by A to obtain $\mathbf{A}^4$ and $\mathbf{A}^5$.

Question 8

8 The curve C has equation $y = \tanh x$ for $x \geqslant 0$.

(a) Sketch C and state the equation of the asymptote. [2]



Examiner comment

- Most candidates drew the correct shape of $y = \tanh x$.
- Graphs from successful candidates were not truncated and got closer to, but did not touch, the asymptote as $x \to \infty$.

(b) By considering a suitable set of N rectangles of unit width, use your sketch to show that

$$\sum_{r=1}^{N} \tanh r > \ln(\cosh N).$$ [3]

$$\sum_{r=1}^{N} \tanh r > \int_0^N \tanh x \, dx$$

$$\int_0^N \tanh x = \Big[\ln\big(\cosh x\big)\Big]_0^N = \ln\big(\cosh N\big)$$

Examiner comment

- Most candidates showed enough detail to justify the given answer, writing $\ln(\cosh x)$ before substituting in the limits 0 and N.
- There was some poor notation, where r was used both in the sum and integral.
- A few candidates used 1 as the lower limit for their integration.

(c) The arc of C joining the point where $x = 0$ to the point where $x = \frac{1}{2}\ln 3$ is rotated through one complete revolution about the x-axis. The area of the surface generated is denoted by S.

(i) Use the substitution $u = \sqrt{1+\operatorname{sech}^4 x}$ to show that

$$S = \pi\int_{\frac{5}{4}}^{\sqrt{2}} \frac{u^2}{u^2-1}\,\mathrm{d}u. \qquad [7]$$

$$S = 2\pi\int_0^{\frac{1}{2}\ln 3} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^{\frac{1}{2}\ln 3} \tanh(x)\sqrt{1+\operatorname{sech}^4(x)}\,dx$$

$$u = \sqrt{1+\operatorname{sech}^4 x} \Rightarrow \frac{du}{dx} = -2\left(1+\operatorname{sech}^4 x\right)^{-\frac{1}{2}}\operatorname{sech}^4 x\tanh x$$

$$x = \tfrac{1}{2}\ln 3 \Rightarrow \cosh x = \tfrac{2}{3}\sqrt{3} \Rightarrow \operatorname{sech} x = \tfrac{1}{2}\sqrt{3} \Rightarrow u = \sqrt{1+\tfrac{9}{16}}$$

$$S = -\pi\int_{\sqrt{2}}^{\frac{5}{4}} u\,\frac{u}{u^2-1}\,du = \pi\int_{\frac{5}{4}}^{\sqrt{2}} \frac{u^2}{u^2-1}\,du$$

Examiner comment

- Most candidates stated the correct formula for the surface area. There were some missing limits and incorrect powers of $\operatorname{sech} x$ and $\tanh x$.
- Most candidates also successfully found $\frac{\mathrm{d}u}{\mathrm{d}x}$. There were a few errors arising from incorrect powers or signs.
- Only the strongest responses fully justified the given limits using the given substitution, showing enough working when substituting $x = \frac{1}{2}\ln 3$.

(ii) Find the exact value of $\pi\int_{\frac{5}{4}}^{\sqrt{2}} \frac{u^2}{u^2-1}\,\mathrm{d}u$. You need not simplify your answer. [3]

$$S = \tfrac{1}{2}\pi\int_{\frac{5}{4}}^{\sqrt{2}} 2 + \frac{1}{u-1} - \frac{1}{u+1}\,du = \pi\left[u + \tfrac{1}{2}\ln(u-1) - \tfrac{1}{2}\ln(u+1)\right]_{\frac{5}{4}}^{\sqrt{2}}$$

$$= \pi\left(\sqrt{2} + \tfrac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) - \tfrac{5}{4} - \tfrac{1}{2}\ln\left(\frac{1}{9}\right)\right) = \pi\left(\sqrt{2} - \tfrac{5}{4} + \ln 3 + \tfrac{1}{2}\ln\left(3-2\sqrt{2}\right)\right)$$

Examiner comment

- There were many fully complete and correct answers seen when finding the given integral.
- A few candidates tried to integrate using $\tanh^{-1}u$, which is invalid due to the restricted domain.

Cambridge Assessment International Education
The Triangle Building, Shaftesbury Road, Cambridge, CB2 8EA, United Kingdom
t: +44 1223 553554
e: info@cambridgeinternational.org www.cambridgeinternational.org