Cambridge Assessment International Education

Example Responses – Paper 1

Cambridge International AS & A Level Further Mathematics 9231

For examination from 2025

Cambridge Pathway

Contents

Introduction....................4
Question 1....................5
Question 2....................7
Question 3....................8
Question 4....................9
Question 5....................11
Question 6....................13
Question 7....................15

Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Further Mathematics 9321.

This booklet contains responses to all questions from June 2025 Paper 14, which have been written by a Cambridge examiner. Responses are accompanied by a brief commentary highlighting common errors and misconceptions where they are relevant.

The question papers and mark schemes are available to download from the School Support Hub

9321 June 2025 Question Paper 14

9321 June 2025 Mark Scheme 14

Past exam resources and other teaching and learning resources are available from the School Support Hub

Question 1

1 **(a)** Use the List of formulae (MF19) to find $\sum_{r=1}^{n}(2r+1)$ in terms of n, simplifying your answer. [2]

$$\sum_{r=1}^{n}(2r+1) = 2\sum_{r=1}^{n} r + n$$

$$= 2\frac{n(n+1)}{2} + n$$

$$= n^2 + 2n$$

Examiner comment

The answer could also be written as $n(n + 2)$.

(b) Show that $\dfrac{2r+1}{(r^2+1)(r^2+2r+2)} = \dfrac{1}{r^2+1} - \dfrac{1}{r^2+2r+2}$. [1]

$$\frac{1}{r^2+1} - \frac{1}{r^2+2r+2} = \frac{r^2+2r+2}{(r^2+1)(r^2+2r+2)} - \frac{r^2+1}{(r^2+1)(r^2+2r+2)}$$

$$= \frac{(r^2+2r+2)-(r^2+1)}{(r^2+1)(r^2+2r+2)}$$

$$= \frac{2r+1}{(r^2+1)(r^2+2r+2)}$$

Examiner comment

- It was common to see an attempt to find the partial fractions. The denominators are quadratic and so the numerators must be taken to be linear.
- The appropriate form to consider is $\dfrac{2r+1}{(r^2+1)(r^2+2r+2)} = \dfrac{Ar+B}{r^2+1} + \dfrac{Cr+D}{r^2+2r+2}$.

(c) Use the method of differences to find $\sum_{r=1}^{n} \frac{2r+1}{(r^2+1)(r^2+2r+2)}$. [2]

$$\sum_{r=1}^{n} \frac{2r+1}{(r^2+1)(r^2+2r+2)} = \sum_{r=1}^{n} \frac{1}{r^2+1} - \frac{1}{(r+1)^2+1}$$

$$= \frac{1}{2} - \frac{1}{5}$$

$$+ \frac{1}{5} - \frac{1}{10}$$

$$+ \frac{1}{10} - \frac{1}{17}$$

....

$$+ \frac{1}{(n-1)^2+1} - \frac{1}{n^2+1}$$

$$+ \frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}$$

$$= \frac{1}{2} - \frac{1}{(n+1)^2+1}$$

Examiner comment

- The answer could also be given as $\frac{1}{2} - \frac{1}{n^2+2n+2}$.
- It is important to show enough complete terms to justify the cancellation. Any term with parts that appear in the final answer should be shown, plus at least one pair of consecutive terms to show how the cancellation works.

(d) Deduce the value of $\sum_{r=1}^{\infty} \frac{2r+1}{(r^2+1)(r^2+2r+2)}$. [1]

as $n \to \infty$ the second term tends to zero, so

$$\sum_{r=1}^{\infty} \frac{2r+1}{(r^2+1)(r^2+2r+2)} = \frac{1}{2}$$

Examiner comment

This part was sometimes omitted.

Question 2

2 Prove by mathematical induction that, for every integer $n \geqslant 2$,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x\ln x) = (-1)^n (n-2)!\,x^{1-n}. \qquad [6]$$

$$\frac{d}{dx}(x\ln x) = x\,\frac{1}{x} + \ln x = 1 + \ln x$$

$$\frac{d^2}{dx^2}(x\ln x) = \frac{1}{x}$$

$(-1)^2(2-2)!\,x^{1-2} = x^{-1}$ so the result holds for $k = 2$

assume that $\frac{d^k}{dx^k}(x\ln x) = (-1)^k(k-2)!\,x^{1-k}$ is true for some integer $k \geqslant 2$

$$\frac{d^{k+1}}{dx^{k+1}}(x\ln x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(x\ln x)\right)$$

$$= \frac{d}{dx}\left((-1)^k(k-2)!\,x^{1-k}\right)$$

$$= (-1)^k(k-2)!\,x^{1-k-1}(1-k)$$

$$= (-1)(-1)^k(k-1)(k-2)!\,x^{1-(k+1)}$$

$$= (-1)^{k+1}((k+1)-2)!\,x^{1-(k+1)}$$

therefore if the result holds for k it also holds for $k = 1$. It is true for $k = 2$ and so by the principle of mathematical induction $\frac{d^n}{dx^n}(x\ln x) = (-1)^n(n-2)!\,x^{1-n}$ for all integers $n \geqslant 2$

Examiner comment

- The two stages for differentiating $x\ln x$ should be clearly shown.
- The substitution of 2 into the formula must be shown to prove it gives the same result as differentiating.
- A clear assumption of the result for some value of n must be shown. Some solutions assumed the result for all values of n.
- The derivative must be arranged to show that it is the same result as substituting $(k + 1)$ into the formula.
- The final statement should include the algebraic form of the result proved and that it is true for all integers $n \geqslant 2$.

Question 3

3 The points A, B and C have position vectors

$$2\mathbf{j}+3\mathbf{k}, \quad -5\mathbf{i}+3\mathbf{j}+\mathbf{k} \quad \text{and} \quad \mathbf{i}+2\mathbf{j}+5\mathbf{k}$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

$\overrightarrow{AB} = -5i + j - 2k \qquad \overrightarrow{AC} = i + 2k$

normal to plane is $(-5i + j - 2k) \times (i + 2k)$

$= (1\times 2 - 2\times 0)i - ((-5)\times 2 - (-2)\times 1)j + ((-5\times 0 - 1\times 1)k)$

$= 2i + 8j - k$

plane is $2x + 8y - z = d$

point A $(0, 2, 3)$ lies on the plane $2(0) + 8(2) - 1(3) = d$ so $d = 13$

$2x + 8y - z = 13$

(b) Find the perpendicular distance from O to the plane ABC. [2]

perpendicular distance from O to the plane is

$$\frac{|2\times 0 + 8\times 0 - 1\times 0 - 13|}{\sqrt{2^2 + 8^2 + 1^2}} = \frac{13}{\sqrt{69}} = 1.57$$

(c) Find the acute angle between the line OA and the plane ABC. [3]

if θ is the angle between the line $\overrightarrow{OA}$ and the normal to the plane ABC

$$\begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \sqrt{69}\sqrt{13}\cos\theta$$

$$\cos\theta = \frac{2(0) + 8(2) - 1(3)}{\sqrt{2^2 + 8^2 + (-1)^2}\sqrt{0^2 + 2^2 + 3^2}} = \sqrt{\frac{13}{69}} = 0.43406$$

$\theta = 64.275°$

acute angle between the line $\overrightarrow{OA}$ and the plane ABC is $90 - 64.275 = 25.7°$

Examiner comment

It was not always clear which angle was being found. It was common to see the final answer $\theta = 64.3°$.

Question 4

4 The cubic equation $x^3+bx^2+cx-1=0$, where b and c are constants, has roots α, β, γ.

It is given that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular.

(a) Show that $\alpha^2+\beta^2+\gamma^2=3$. [4]

$x^3+bx^2+cx-1=0$

$\begin{vmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{vmatrix} = 0$

$1(1-\gamma^2)-\alpha(\alpha-\beta\gamma)+\beta(\alpha\gamma-1)=0$

$1-\alpha^2-\beta^2-\gamma^2+\alpha\beta\gamma+\alpha\beta\gamma=0$

from the given equation $\alpha\beta\gamma = \dfrac{-(-1)}{1} = 1$

so $1-(\alpha^2+\beta^2+\gamma^2)+2=0$

$\alpha^2+\beta^2+\gamma^2=3$

Examiner comment

Candidates needed to clearly refer to the coefficients of the given equation to prove that $\alpha\beta\gamma=1$.

(b) It is given that $\alpha^3+\beta^3+\gamma^3=3$ and that the constants b and c are positive.

Find the values of b and c. [6]

$$\alpha^3+b\alpha^2+c\alpha-1=0$$

$$\beta^3+b\beta^2+c\beta-1=0$$

$$\gamma^3+b\gamma^2+c\gamma-1=0$$

summing $\left(\alpha^3+\beta^3+\gamma^3\right)+b\left(\alpha^2+\beta^2+\gamma^2\right)+c\left(\alpha+\beta+\gamma\right)-3=0$

using $\alpha^2+\beta^2+\gamma^2=3$ (part (a))

$\alpha+\beta+\gamma=-b$ (from original equation)

$\alpha^3+\beta^3+\gamma^3=3$ (given)

$$3+3b-bc-3=0$$

$$b\left(3-c\right)=0$$

$b>0$ so $c=3$

$$\left(\alpha+\beta+\gamma\right)^2=\left(\alpha^2+\beta^2+\gamma^2\right)+2\left(\alpha\beta+\beta\gamma+\alpha\gamma\right)$$

$$\left(-b\right)^2=3+2c=9$$

$$b>0 \text{ so } b=3$$

Examiner comment

Some candidates did not use the condition that both b and c are positive, and gave extra answers.

Question 5

5 The matrix **M** represents a sequence of two transformations in the x-y plane given by a one-way stretch in the x-direction, scale factor 3, followed by a reflection in the line $y = x$.

(a) Find **M**. [3]

one-way stretch in the x direction with scale factor 3

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

reflection in y = x

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$$

Examiner comment

The transformation applied first must be placed on the right-hand side.

(b) Give full details of the geometrical transformation in the x-y plane represented by $\mathbf{M}^{-1}$. [3]

reflection in y = x

followed by one-way stretch in the x-direction with scale factor $\frac{1}{3}$

Examiner comment

- The order of transformations was reversed.
- The transformations needed to be described in the same way as in part **(a)**.
- Candidates were not asked to calculate $\mathbf{M}^{-1}$ at this point.

The matrix **N** is such that $\mathbf{MN} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

(c) Find **N**. [3]

$$MN = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

$$M^{-1}MN = M^{-1}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

$$N = M^{-1}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

$$\text{Det } \mathbf{M} = 0 \times 0 - 1 \times 3 = -3$$

$$\mathbf{M}^{-1} = \frac{1}{-3}\begin{pmatrix} 0 & -1 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{2}{3} \\ 1 & 2 \end{pmatrix}$$

Examiner comment

The equation mustbe left-multiplied by $\mathbf{M}^{-1}$.

(d) Find the equations of the invariant lines, through the origin, of the transformation in the *x*-*y* plane represented by **MN**. [5]

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 2y \end{pmatrix}$$

$$3x + 2mx = m(x + 2mx)$$

$$3 + 2m = m(1 + 2m) \Rightarrow 2m^2 - m - 3 = 0$$

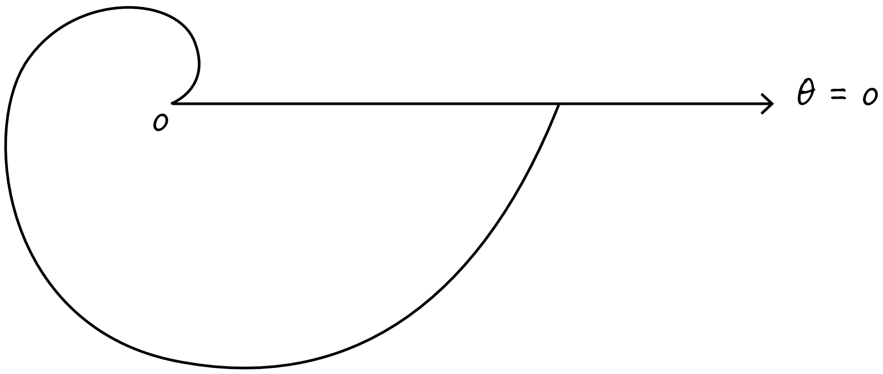
$$m = \frac{3}{2} m = -1$$

$$y = \frac{3}{2}x \text{ and } y = -x$$

Question 6

6 The curve C has polar equation $r = a\tan\left(\frac{1}{8}\theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant 2\pi$.

(a) Sketch C and state, in terms of a, the greatest distance of a point on C from the pole. [3]



greatest distance of a point on C from the pole is a

Examiner comment

- The pole and the initial line needed to be clearly identified.
- Many graphs had places where r decreased as θ increased. $r = \tan\left(\frac{\theta}{8}\right)$ is an increasing function on $[0, 2\pi]$ and the graph needed to show this.
- Marking a on the graph or stating $(a, 2\pi)$ did not identify the maximum distance from the pole. The answer needed to be clearly identified as the maximum distance.

(b) Find, in terms of a, the area of the region bounded by C and the initial line. [4]

$$\text{Area} = \frac{1}{2}a^2\int_0^{2\pi}\tan^2\left(\frac{1}{8}\theta\right)d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi}\sec^2\left(\frac{1}{8}\theta\right) - 1\ d\theta$$

$$= \frac{1}{2}a^2\left[8\tan\left(\frac{1}{8}\theta\right) - \theta\right]_0^{2\pi}$$

$$= \frac{1}{2}a^2\left(\left(8\tan\left(\frac{1}{8}2\pi\right) - 2\pi\right) - \left(8\tan 0 - 0\right)\right)$$

$$= \frac{1}{2}a^2\left(8 - 2\pi\right) = a^2\left(4 - \pi\right)$$

Examiner comment

- The limits needed to be clearly shown.
- Both limits needed to be substituted into the integrated function.
- The integration could be carried out over the single range [0,2π]. Several solutions divided it into two parts, which gave more chance of making a numerical error.

(c) Show that, at the point on C furthest from the initial line,

$$4\sin\left(\tfrac{1}{4}\theta\right)\cos\theta + \sin\theta = 0$$

and verify that this equation has a root between 4.95 and 5. [6]

when furthest from initial line $y = r\sin\theta = a\sin\theta\tan\left(\frac{1}{8}\theta\right)$ is maximum.

$$\frac{dy}{d\theta} = a\cos\theta\tan\left(\frac{1}{8}\theta\right) + \frac{1}{8}a\sin\theta\sec^2\left(\frac{1}{8}\theta\right)$$

when $\dfrac{dy}{d\theta} = 0$

$$a\cos\theta\tan\left(\frac{1}{8}\theta\right) + \frac{1}{8}a\sin\theta\sec^2\left(\frac{1}{8}\theta\right) = 0$$

$$\cos\theta\sin\left(\frac{1}{8}\theta\right)\cos\left(\frac{1}{8}\theta\right) + \frac{1}{8}\sin\theta = 0$$

$\cos\left(\theta\right) \neq 0$

$$8\sin\left(\frac{1}{8}\theta\right)\cos\left(\frac{1}{8}\theta\right) + \tan\theta = 0$$

since $2\sin\left(\frac{1}{8}\theta\right)\cos\left(\frac{1}{8}\theta\right) = \sin\left(\frac{1}{4}\theta\right)$

$$4\sin\left(\frac{1}{4}\theta\right) + \tan\theta = 0$$

when $\theta = 4.95$, $\quad 4\sin\left(\frac{1}{4}\theta\right) + \tan\theta = -0.349\ldots < 0$

when $\theta = 5$, $\quad 4\sin\left(\frac{1}{4}\theta\right) + \tan\theta = 0.415\ldots > 0$

There is a change of sign so the equation has a root between 4.95 and 5.

Examiner comment

- When dividing by a factor, candidates should always check whether it could be zero.
- Some candidates did not show the double angle formula that they were using. They were proving a given result so all details were needed.
- The numerical values of the function $4\sin\left(\frac{1}{4}\theta\right) + \tan\theta$ needed to be calculated to show the sign change.

Question 7

7 The curve C has equation $y = \dfrac{x^2+x-4}{x^2+x+2}$.

(a) State the equation of the asymptote of C. [1]

$y = 1$

Examiner comment

The answer needed to be given as an equation.

(b) Show that, for all real values of x, $-\frac{17}{7} \leqslant y < 1$. [4]

discriminant method

$$y = \frac{x^2 + x - 4}{x^2 + x + 2}$$

rearranging $x^2(y-1) + x(y-1) + 2y + 4 = 0$

for this equation to have real roots the discriminant $\geqslant 0$

$$(y-1)^2 - 4(y-1)(2y+4) \geqslant 0$$

$$(y-1)(y-1-8y-16) \geqslant 0$$

$$(y-1)(7y+17) \leqslant 0$$

$$-\frac{17}{7} \leqslant y \leqslant 1$$

$y = 1 \Rightarrow x^2 + x - 4 = x^2 + x - 2 \Rightarrow -4 = -2$ which is impossible

so $y \neq 1$ (the asymptote)

$$-\frac{17}{7} \leqslant y \leqslant 1$$

inequality method $y = \dfrac{x^2 + x - 4}{x^2 + x + 2}$

rearranging $y = 1 - \dfrac{6}{x^2 + x + 2}$

$$y = 1 - \frac{6}{\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}}$$

since $\left(x+\frac{1}{2}\right)^2+\frac{7}{4}\geqslant\frac{7}{4}$

$$\frac{6}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}\leqslant 6\div\frac{7}{4}=\frac{24}{7}$$

$$y\geqslant 1-\frac{24}{7}$$

$$y\geqslant -\frac{17}{7}$$

also

$$\left(x+\frac{1}{2}\right)^2+\frac{7}{4}>0$$

$$\frac{6}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}>0$$

$$y=1-\frac{6}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}<1$$

$$-\frac{17}{7}\leqslant y<1$$

Examiner comment

- The answer was given in the question, so full working and reasoning needed to be shown.
- Most candidates found the numerical values given in the question, but did not give a full explanation of why the inequality was given in this form.
- For the discriminant method, it needed to be clear that we were looking for real roots, and the correct inequality must be used.
- Factors should be shown to justify the regions chosen.
- Candidates needed to give an explanation as to why $y\neq 1$.
- For the inequality method, completing the square in the denominator proves the range of values for $\left(x+\frac{1}{2}\right)^2+\frac{7}{4}$ and justifies the subsequent rearrangements to find the range for y.

(c) Find the coordinates of any stationary points of C. [3]

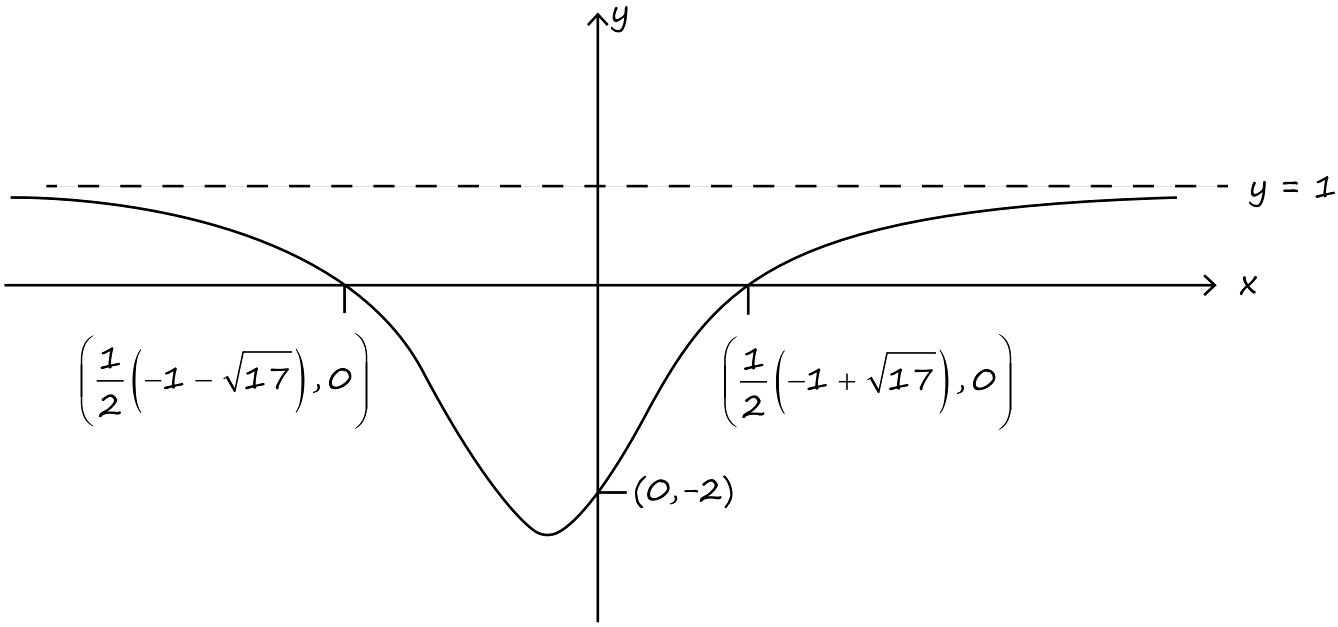
$$\frac{dy}{dx} = \frac{(x^2 + x + 2)(2x + 1) - (x^2 + x - 4)(2x + 1)}{(x^2 + x + 2)^2}$$

$$= \frac{6(2x + 1)}{(x^2 + x + 2)^2}$$

for stationary point $\frac{dy}{dx} = 0$ so $2x + 1 = 0 \quad x = -\frac{1}{2}$

stationary point is $\left(-\frac{1}{2}, -\frac{17}{7}\right)$

(d) Sketch C, stating the coordinates of the intersections with the axes. [3]



when $x = 0 \quad y = \frac{0 + 0 - 4}{0 + 0 + 2} = -2$

when $y = 0 \quad x^2 + x - 4 = 0 \quad x = \frac{-1 \pm \sqrt{17}}{2}$

intersections with the axes are

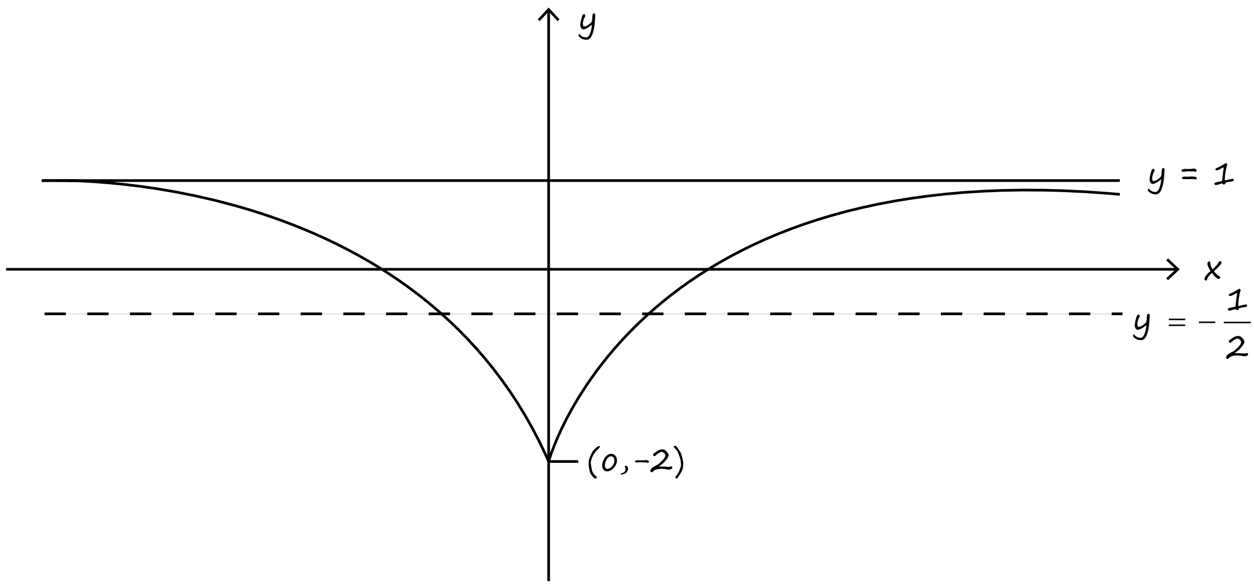
$(0, -2)$, $\left(\frac{1}{2}\left(-1 + \sqrt{17}\right), 0\right)$, $\left(\frac{1}{2}\left(-1 - \sqrt{17}\right), 0\right)$

Examiner comment

- Axes and asymptote needed to be labelled.
- The graph needed to show a smooth minimum point.
- The graph needed to approach the asymptote steadily and closely.

(e) Sketch the graph with equation $y = \dfrac{|x|^2 + |x| - 4}{|x|^2 + |x| + 2}$ and find the set of values of x for which

$\dfrac{|x|^2 + |x| - 4}{|x|^2 + |x| + 2} < -\frac{1}{2}$. [5]



$$\frac{|x|^2 + |x| - 4}{|x|^2 + |x| + 2} = -\frac{1}{2}$$

$$2\left(|x|^2 + |x| - 4\right) = -1\left(|x|^2 + |x| + 2\right)$$

$$3|x|^2 + 3|x| - 6 = 0$$

$$\left(|x| - 1\right)\left(|x| + 2\right) = 0$$

$$|x| \geq 0$$

$$|x| = 1$$

$$x = \pm 1$$

using the graph

$$\frac{|x|^2 + |x| - 4}{|x|^2 + |x| + 2} < -\frac{1}{2} \text{ when } -1 < x < 1$$

Examiner comment

- Another method was to consider the right-hand side of the graph, where $x \geqslant 0$ to find the value $x = 1$, then the symmetry of the graph gave the final answer.
- The graph needed to show a sharp change of direction at the point (0, -2) and symmetry in the y-axis.

Cambridge Assessment International Education
The Triangle Building, Shaftesbury Road, Cambridge, CB2 8EA, United Kingdom
t: +44 1223 553554
e: info@cambridgeinternational.org   www.cambridgeinternational.org