Cambridge Assessment
International Education

Example Responses – Paper 4

# Cambridge International AS & A Level Further Mathematics 9231

For examination from 2025

Cambridge Pathway

# Contents

# Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Further Mathematics 9321.

This booklet contains responses to all questions from June 2025 Paper 44, which have been written by a Cambridge examiner. Responses are accompanied by a brief commentary highlighting common errors and misconceptions where they are relevant.

The question papers and mark schemes are available to download from the School Support Hub

**9321 June 2025 Question Paper 44**

**9321 June 2025 Mark Scheme 44**

Past exam resources and other teaching and learning resources are available from the School Support Hub

# Question 1

**1** A random sample of 12 observations of a normal random variable is taken. The results give unbiased estimates for the population mean and variance as 10.24 and 0.52 respectively.

Test, at the 10% significance level, the null hypothesis that the population mean is 10.6 against the alternative hypothesis that the population mean is less than 10.6. [4]

$H_0 : \mu = 10.6$, $H_1 : \mu < 10.6$

$$t = \frac{10.24 - 10.6}{\sqrt{\frac{0.52}{12}}} = -1.729$$

critical value is −1.363.

Since -1.729 < -1.363, the observed result is significant, and we reject the null hypothesis.

There is sufficient evidence to support that the population mean is less than 10.6.

## Examiner comment

- Errors were seen in the denominator of the test statistic. Some candidates attempted to derive $s_x^2$ not realising that this is the unbiased estimate of the population variance stated in the question. Denominators of $\sqrt{0.52}$ and $\frac{0.52}{\sqrt{12}}$ were also seen.
- Some candidates did not handle the negative test statistic correctly, either by writing -1.729 > -1.363 or by stating that -1.729 < -1.363 was not significant. Signs needed to be consistent for the test statistic and the critical value to make a valid informed comparison.

# Question 2

**2** The level of sound produced by a particular type of machine was measured for a random sample of 11 such machines. The results, in suitable units, are shown below.

| Machine | *A* | *B* | *C* | *D* | *E* | *F* | *G* | *H* | *I* | *J* | *K* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sound level | 7.66 | 8.48 | 8.21 | 7.98 | 8.01 | 7.77 | 8.25 | 8.11 | 8.03 | 8.16 | 7.92 |

**(a)** Use a Wilcoxon signed-rank test to test whether the average sound level produced by this type of machine is more than 8.00. Use a 5% significance level. [6]

$H_0$ : population median = 8.00

$H_1$ : population median > 8.00

| differences | −0.34 | 0.48 | 0.21 | −0.02 | 0.01 | −0.23 | 0.25 | 0.11 | 0.03 | 0.16 | −0.08 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ranks | −10 | 11 | 7 | −2 | 1 | −8 | 9 | 5 | 3 | 6 | −4 |

Q = 24, P = 42, so the test statistic is 24.

critical value is 13.

as 24 > 13, the observed result is not significant, do not reject $H_0$.

there is insufficient evidence to support that the average sound level is more than 8.00.

**(b)** Give a reason why a Wilcoxon signed-rank test may be more appropriate than a *t*-test in this case. [1]

It is not known if the population is normally distributed.

## Examiner comment

- In part **(a)**, a common error was not specifying population when forming hypotheses or referring to the mean or $\mu$, or simply using the word ‘average’. The conclusion in context should refer to average sound level. When accepting the null hypothesis, candidates needed to understand that this does not indicate evidence for the null hypothesis, so conclusions indicating that the average sound level is 8.00 were not awarded any marks.
- In part **(b)**, some candidates referred to the data being normal, or referred to the population of differences which would only apply to a 2-sample test.

# Question 3

**3** A shop selling electrical goods has a team of three salespeople: Avril, Ben and Charlie. The manager wishes to investigate whether the salespeople are equally successful at selling particular types of items. The following table gives a record of a random sample of 250 sales of laptops, cameras and televisions, with the number sold by each of the three salespeople.

| | Type of item | | | |
|---|---|---|---|---|
| | Laptop | Camera | Television | Total |
| Avril | 31 | 40 | 24 | 95 |
| Ben | 23 | 45 | 29 | 97 |
| Charlie | 21 | 25 | 12 | 58 |
| Total | 75 | 110 | 65 | 250 |

Test, at the 10% significance level, whether there is independence between the type of item sold and the salesperson. [7]

$H_0$ : type of item sold is independent of salesperson

$H_1$ : type of item sold is not independent of salesperson

Observed and expected frequencies are shown in the following table.

| | Laptop | | Camera | | Television | |
|---|---|---|---|---|---|---|
| Avril | Obs | Exp | Obs | Exp | Obs | Exp |
| Ben | 31 | 28.5 | 40 | 41.8 | 24 | 24.7 |
| Charlie | 23 | 29.1 | 45 | 42.68 | 29 | 25.22 |
| | 21 | 17.4 | 25 | 25.52 | 12 | 15.08 |

$$\chi^2 = \frac{(31-28.5)^2}{28.5} + \frac{(40-41.8)^2}{41.8} + \frac{(24-24.7)^2}{24.7} + \frac{(23-29.1)^2}{29.1} + \frac{(45-42.68)^2}{42.68}$$

$$+ \frac{(29-25.22)^2}{25.22} + \frac{(21-17.4)^2}{17.4} + \frac{(25-25.52)^2}{25.52} + \frac{(12-15.08)^2}{15.08}$$

$$= 3.67 \text{ (3 sf)}$$

as $3.67 < 7.779$, the observed result is not significant, do not reject $H_0$

insufficient evidence to suggest that type of item sold and salesperson are not independent

## Examiner comment

- A common error was not fully specifying the hypotheses, often by omitting the word ‘sold’. The expected frequencies needed to be seen, and at least two correct expressions for the calculation of the test statistic needed to be presented so that if the candidate made a slip with their calculation, a method mark could still be awarded.
- It was common to see a conclusion in context, following correct working, state that the result suggested independence. As stated in the comment for Question 2, when accepting the null hypothesis, candidates should understand that this does not indicate evidence for the null hypothesis.

# Question 4

**4** The continuous random variable $X$ has probability density function f given by

$$\mathrm{f}(x) = \begin{cases} kx & 0 \leqslant x < 1, \\ kx^2 & 1 \leqslant x \leqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

**(a)** Show that $k = \frac{6}{17}$. [2]

$$\int_0^1 kx\,dx + \int_1^2 kx^2\,dx = 1$$

$$\left[\frac{1}{2}kx^2\right]_0^1 + \left[\frac{1}{3}kx^3\right]_1^2 = 1$$

$$\frac{1}{2}k + \frac{1}{3}k(8-1) = 1$$

$$\frac{17}{6}k = 1$$

$$k = \frac{6}{17}$$

**(b)** Find the cumulative distribution function of $X$. [3]

for $0 \leqslant x < 1$:

$$F(x) = \int_0^x \frac{6}{17}t\,dt$$

$$= \left[\frac{3}{17}t^2\right]_0^x$$

$$= \frac{3}{17}x^2$$

for $1 \leqslant x \leqslant 2$:

$$F(x) = F(1) + \int_1^x \frac{6}{17}t^2\,dt$$

$$= \frac{3}{17} + \left[\frac{2}{17}t^3\right]_1^x$$

$$= \frac{3}{17} + \frac{2}{17}x^3 - \frac{2}{17}$$

$$= \frac{2}{17}x^3 + \frac{1}{17}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{17}x^2 & 0 \leqslant x < 1 \\ \frac{2}{17}x^3 + \frac{1}{17} & 1 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

**(c)** Find the median value of $X$. [2]

$F(1) = \frac{3}{17} < \frac{1}{2}$, and so $1 < m < 2$.

$$\frac{2}{17}m^3 + \frac{1}{17} = \frac{1}{2}$$

$$m^3 = \frac{15}{4}$$

$$m = \sqrt[3]{\frac{15}{4}}$$

$$= 1.55 \ (3\,\text{sf})$$

**(d)** Find $\mathrm{E}\left(\frac{1}{X}\right)$. [2]

$$E\left(\frac{1}{X}\right) = \int_0^1 \frac{6}{17}x\left(\frac{1}{x}\right)dx + \int_1^2 \frac{6}{17}x^2\left(\frac{1}{x}\right)dx$$

$$= \frac{6}{17}\left[x\right]_0^1 + \frac{6}{17}\left[\frac{1}{2}x^2\right]_1^2$$

$$= \frac{6}{17}(1-0) + \frac{6}{17}\left(2 - \frac{1}{2}\right)$$

$$= \frac{15}{17}$$

## Examiner comment

- In part **(a)**, the answer was given, so full working was required to provide a convincing solution.
- In part **(b)**, the most common error was in finding the constant $\frac{1}{17}$. There was often no attempt to find this constant, or the value $\frac{3}{17}$ was used.
- Many candidates were successful in part **(c)**. Some started with the probability density function, essentially reproducing their working for part **(b)**, for example by forming the equation $\int_0^1 \frac{6}{17}x\,\mathrm{d}x + \int_1^m \frac{6}{17}x^2\,\mathrm{d}x = \frac{1}{2}$.
- Some candidates attempted to integrate the CDF instead of the PDF.

# Question 5

**5** Eric has three identical coins, each of which is biased so that the probability of obtaining a head when it is thrown is $\frac{1}{3}$. The random variable $X$ is the number of heads obtained when Eric throws the three coins at the same time.

**(a)** Find the probability generating function $\mathrm{G}_X(t)$ of $X$. [2]

$$P(X=0)=\left(\frac{2}{3}\right)^3=\frac{8}{27}$$

$$P(X=1)=3\times\left(\frac{2}{3}\right)^3\times\left(\frac{1}{3}\right)=\frac{12}{27}$$

$$P(X=2)=3\times\left(\frac{2}{3}\right)^2\times\left(\frac{1}{3}\right)^2=\frac{6}{27}$$

$$P(X=3)=\left(\frac{1}{3}\right)^3=\frac{1}{27}$$

$$G_X(t)=\frac{8}{27}+\frac{12}{27}t+\frac{6}{27}t^2+\frac{1}{27}t^3$$

Eric also has two fair 6-sided dice with faces numbered 1 to 6. The random variable $Y$ is the number of sixes obtained when Eric throws the two dice at the same time. It is given that the probability generating function of $Y$ is $\frac{25}{36}+\frac{10}{36}t+\frac{1}{36}t^2$.

Eric throws the three coins and the two dice. The random variable $Z$ is the sum of the number of heads obtained and the number of sixes obtained.

**(b)** Find the probability generating function $\mathrm{G}_Z(t)$ of $Z$, expressing your answer as a polynomial in $t$. [3]

$$G_Z(t)=G_X(t)\times G_Y(t)$$

$$=\left(\frac{8}{27}+\frac{12}{27}t+\frac{6}{27}t^2+\frac{1}{27}t^3\right)\left(\frac{25}{36}+\frac{10}{36}t+\frac{1}{36}t^2\right)$$

$$=\frac{200}{972}+\frac{80}{972}t+\frac{300}{972}t+\frac{8}{972}t^2+\frac{120}{972}t^2+\frac{150}{972}t^2+\frac{12}{972}t^3+\frac{60}{972}t^3$$

$$+\frac{25}{972}t^3+\frac{6}{972}t^4+\frac{10}{972}t^4+\frac{1}{972}t^5$$

$$=\frac{1}{972}\left(200+380t+278t^2+97t^3+16t^4+t^5\right)$$

**(c)** Use $\mathrm{G}_Z(t)$ to find $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$. [5]

$$G_Z'(t) = \frac{1}{972}\left(380 + 556t + 291t^2 + 64t^3 + 5t^4\right)$$

$$E(Z) = G_Z'(1) = \frac{1296}{972} = \frac{4}{3}$$

$$G_Z''(t) = \frac{1}{972}\left(556 + 582t + 192t^2 + 20t^3\right)$$

$$G_Z''(1) = \frac{1350}{972} = \frac{25}{8}$$

$$Var(Z) = G_Z''(1) + G_Z'(1) - \left(G_Z'(1)\right)^2$$

$$= \frac{25}{18} + \frac{4}{3} - \left(\frac{4}{3}\right)^2$$

$$= \frac{17}{18}$$

## Examiner comment

- In part **(a)**, marks were awarded for any correct coefficients which did not need to be simplified.
- In part **(b)**, most errors were numerical, often following reduction of fractions. Candidates who checked that $\mathrm{G}_Z(1) = 1$ were almost always successful.
- In part **(c)**, the forms of $\mathrm{G}'_Z(t)$ and $\mathrm{G}'_Z(t)$ may be inferred from expressions for $\mathrm{G}'_Z(1)$ and $\mathrm{G}''_Z(1)$ such as $\mathrm{G}'_Z(1) = \frac{1}{972}(380 + 556 + 291 + 64 + 5)$ and $\mathrm{G}'_Z(1) = \mathrm{G}''_Z(1) = \frac{1}{972}(556 + 582 + 192 + 20)$.

# Question 6

**6** Lina and Mona are two statisticians who also write songs. The 'time' of a song is the number of minutes for which it lasts. For a random sample of 10 of her songs, Lina calculates a 95% confidence interval for the population mean time, $\mu$ minutes. This confidence interval is $2.95 \leqslant \mu \leqslant 3.13$. The times, $x$ minutes, of Lina's songs are normally distributed.

**(a)** Find the values of $\sum x$ and $\sum x^2$ for the 10 songs in Lina's sample. [5]

$\bar{x} = \frac{1}{2}(3.13 + 2.95) = 3.04$

therefore, $\sum x = 10 \times 3.04 = 30.4$

$$3.04 - 2.262 \times \frac{s_x}{\sqrt{10}} = 2.95$$

$$s_x = \frac{\sqrt{10}(3.04 - 2.95)}{2.262}$$

$$= 0.12582$$

$$\frac{1}{9}\left(\sum x^2 - \frac{1}{10}(30.4^2)\right) = 0.12582^2$$

$$\sum x^2 = 9 \times 0.12582^2 + \frac{1}{10}(30.4^2)$$

$$= 92.6(3 \text{ sf})$$

Mona's songs have times, $y$ minutes, that are normally distributed. The times for a random sample of 8 of Mona's songs are summarised as follows.

$$\Sigma y = 24.8 \qquad \Sigma y^2 = 76.98$$

Mona claims that the population mean time of her songs is greater than the population mean time of Lina's songs.

**(b)** Assuming that the two distributions have the same population variance, test at the 5% significance level whether there is evidence to support Mona's claim. [8]

$H_0 : \mu_Y = \mu_X , \quad H_1 : \mu_Y > \mu_X$

$s_x^2 = 0.12582^2 = 0.01583$

$s_y^2 = \frac{1}{7}\left(76.98 - \frac{1}{8}\left(24.8^2\right)\right) = \frac{1}{70}$

$s_p^2 = \dfrac{9 \times 0.12582^2 + 7 \times \frac{1}{70}}{10 + 8 - 2} = 0.015155$

$t = \dfrac{3.04 - 3.1}{\sqrt{0.015155\left(\frac{1}{10} + \frac{1}{8}\right)}} = -1.028$

critical value is -1.746

since -1.028 > -1.746, the observed result is not significant, do not reject H0

insufficient evidence to support Mona's claim

## Examiner comment

- In part **(a)**, the most common mistake was in choosing the correct critical value. Often an adjacent tabular value was chosen, or even a $z$-value.
- In part **(b)**, the most common error was to combine the sample variances rather than pool these estimates assuming that the population variances are equal as stated in the question.
- Some candidates used their answers to part **(a)** to calculate $s_x^2$ rather than square the value for $s_x$ that they used to compute these values. This often led to a loss of accuracy particularly when using the rounded value for $\Sigma x^2$ of 92.6.

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